Mixed Hodge Structures

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Introduction

In 1949 A. Weil [W] conjectured the existence of remarkable relations between the cohomological structure of algebraic varieties over the complex numbers and the Diophantine structure of algebraic varieties over finite fields. One of these conjectures was the Riemann hypothesis for varieties over a finite field. P. Deligne proved the Weil conjectures in 1974, a result for which he was awarded a Fields Medal in 1978 (see [K]). This result was an achievement of the new foundation of algebraic geometry begun by A. Grothendieck and his school in the 1960's.

Grothendieck had foreseen in the 1960's that establishing the Weil conjectures for projective non-singular varieties over finite fields would have had important consequences for the cohomological structure of arbitrary varieties over the complex numbers, via the comparison of Artin-Grothendieck cohomology and classical cohomology. This was formulated in his "Yoga of weights" and his conjectural theory of motives, which intends to unify all cohomology theories. Deligne developed this philosophy concerning cohomology theories for algebraic varieties in his visionary article *Théorie de Hodge I* ([HI]), published in 1970. He gives there a heuristic comparison dictionary between statements in ℓ-adic cohomology and statements in Hodge theory, and explains how mixed Hodge structures over $\mathbb{C}$ are the analogue of Galois modules, endowed with the weight filtration defined by the magnitude of the eigenvalues of Frobenius endomorphism, which appears in ℓ-adic cohomology. This leads to the expectation that the cohomology of every algebraic variety over $\mathbb{C}$ carries a natural mixed Hodge structure.

A mixed Hodge structure consists of data (H, W, F) where H is a finitely generated abelian group, the weight filtration W is an increasing filtration of $H \otimes \mathbb{Q}$, and the Hodge filtration F is a decreasing filtration of $H \otimes \mathbb{C}$ such that F induces for each n a decomposition $\mathrm{Gr}^n_W(H \otimes \mathbb{C}) = \oplus_{p+q=n} H^{p,q}$ with $\overline{H^{p,q}} = H^{q,p}$. By classical Hodge theory, if X is a projective (or more generally a compact Kähler) manifold, then the cohomology $H^i(X, \mathbb{Z})$ carries a Hodge structure which is pure of weight i, i.e.,with $W_j = 0$ for $j < i$ and $W_i = H^i(X, \mathbb{Q})$. The morphisms of mixed Hodge structures are asked to be compatible with the weight filtration and the Hodge filtration. Deligne shows that mixed Hodge structures form an abelian category. Moreover, he proves that the weight filtration and the Hodge filtration are extremely stable, in the sense that both are strictly preserved under morphisms (see [HII]).

In the late 1960's and early 1970's Deligne developed independently of the Weil conjectures the complete theory of the weight filtration for complex algebraic varieties. He achieved this by making systematic use of Hironaka's resolution of singularities, of the theory of differential forms with logarithmic poles and his own theory of cohomological descent (see [HI], [HII] and [HIII]). The main result of Deligne's fundamental article *Théorie de Hodge III* ([HIII]) is the construction of a natural mixed Hodge structure on the cohomology of any algebraic variety over $\mathbb{C}$. Moreover any morphism of algebraic varieties induces a morphism of mixed Hodge structures. Roughly speaking, the weight filtration expresses how the cohomology of a complex algebraic variety can be built up in terms of the cohomology of non-singular projective varieties. The mixed Hodge theory of Deligne should be seen as a far-reaching generalization of the classical theory of "differentials of the second kind" (see [HII]) as well as Hodge theory.

The key point for endowing cohomology groups $H^\bullet$ with a mixed Hodge structure is to construct a spectral sequence E which abuts to $H^\bullet$, such that each $E_r^{p,q}$ carries a natural pure Hodge structure of weight q and all differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are morphisms of Hodge structures. The vanishing of d_r for $r \geq 2$ is a consequence of the following elementary fact (see the "scholie" in [HI] p. 427): Let H and H' be Hodge structures of weight n and n', with $n > n'$, and let $f : H \to H'$ be a homomorphism such that $f : H \otimes \mathbb{C} \to H' \otimes \mathbb{C}$ respects the Hodge filtration F; then $f = 0$. The weight filtration on $H^\bullet$ is then defined as that of the abutment of E. This program is achieved via the concept of (mixed) Hodge complex, introduced by Deligne in [HIII]. It is motivated by an algebraic reformulation of the Hodge decomposition for compact Kähler manifolds in terms of the hypercohomology of some complexes of sheaves. The core of the theory of Deligne is in [HII]. There he develops homological algebra techniques dealing with a complex endowed with two filtrations and several spectral sequences naturally associated to this situation. This study reveals abstract properties ensuring that several hypercohomology groups of a mixed Hodge complex of sheaves yield a mixed Hodge structure. This is expressed formally in the Fundamental Theorem (Theorem 4.10 in this survey). As a consequence, we have the following crucial general principle: In order to get a mixed Hodge structure on hypercohomology of some geometric object, it suffices to construct a suitable mixed Hodge complex of sheaves on this object. This principle has been applied with a lot of success in the 1970's by Steenbrink ([St1],[St2]), to endow the groups of vanishing cycles with a mixed Hodge structure. In the local case, it leads to powerful invariants of hypersurfaces with isolated singularities: for instance, the spectrum of the singularity (see [St2]).

The present survey contains four sections.

The formal aspects of Hodge theory, namely the definitions of Hodge structures and mixed Hodge structures, are treated in Section 1.

Section 2 introduces varieties with normal crossings whose components are compact Kähler manifolds. They are the simplest example of singular varieties that carry a natural mixed Hodge structure, coming from the Hodge structures of each component. We follow [GS] and present ideas of the general theory in an "elementary" way that tries to stay as close as possible to the spirit of classical Hodge theory. The constant complex sheaf is replaced by a double complex, associated to the combinatorics of a variety with normal crossings, from which the weight and the Hodge filtrations are easily deduced. To link the cohomology of the total double complex with the cohomology of its rows, a powerful algebraic tool is needed: the spectral sequence of a filtered complex, which is described in detail.

Section 3 contains a detailed exposition of the construction of a natural mixed Hodge structure on the cohomology of a smooth quasi-projective algebraic variety, as given by Deligne in [HII]. The construction, essentially algebraic, is based on one hand on Hodge theory, and on the other hand on Hironaka's resolution of singularities. In this way it is possible to express the cohomology of a smooth quasi-projective variety in terms of the cohomology of smooth projective varieties, by means of spectral sequences.

In Section 4 we give the proof of the Fundamental Theorem of Deligne. It is a consequence of sophisticated facts concerning a complex endowed with two filtrations when considering several spectral sequences.

1 Hodge structures

1.1 Pure Hodge structures

The prototype of a Hodge structure is the cohomology of a compact Kähler manifold. A Kähler manifold is a complex hermitian manifold such that the associated metric form is closed. Examples are given by any projective manifold equipped with its Fubini-Study metric. The condition on the metric has deep consequences on the geometry of the manifold. If X is a compact Kähler manifold and $H^{p,q}(X)$ is the space of cohomology classes of differential forms whose harmonic representative is of type (p,q), then the *Hodge decomposition theorem* (see e.g. [GH]) gives the following direct sum decomposition of the de Rham cohomology:

$$H^n_{DR}(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

and moreover $H^{q,p}(X) = \overline{H^{p,q}(X)}$. According to the formalism of Deligne ([HII], [HIII]) this is a Hodge structure of weight n.

Definition 1.1. Let $H_{\mathbb{R}}$ be a $\mathbb{R}$-vector space of finite dimension over $\mathbb{R}$, denote by $H = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $H_{\mathbb{R}}$.
A *Hodge structure of weight* n on H is a given lattice $H_{\mathbb{Z}} \subset H_{\mathbb{C}}$ plus a decomposition $H = \bigoplus_{p+q=n} H^{p,q}$ with $H^{q,p} = \overline{H^{p,q}}$.

By convention $H^{p,q} = 0$ if $p + q \neq n$.

Remark 1.2. The lattice can be replaced by a $\mathbb{Q}$- (resp. $\mathbb{R}$-) vector space. In this case we have a rational (resp. a real) Hodge structure on H. In the case of a real Hodge structure the only condition to be fulfilled is on the decomposition. In the definition we can ask that $H_{\mathbb{Z}}$ is an abelian group and H its complexification. This is only apparently more general, because torsion is not important in the study of Hodge structures.

Definition 1.3. A map $\varphi : H \longrightarrow H'$ between two complex spaces endowed with Hodge structure is a *morphism of Hodge structures* if $\varphi(H_{\mathbb{Z}}) \subset H'_{\mathbb{Z}}$ and $\varphi(H^{p,q}) \subset H'^{p,q}\ \forall p, q$.

Definition 1.4. A map $\varphi : H \longrightarrow H'$ between two complex spaces endowed with Hodge structure is a *morphism of Hodge structures of type* (r, r) if $\varphi(H_{\mathbb{Z}}) \subset H'_{\mathbb{Z}}$ and $\varphi(H^{p,q}) \subset H'^{p+r,q+r}\ \forall p, q$.

If the weight of H' does not equal the weight of H plus $2r$, the only morphism of type (r, r) is the 0 morphism.

There is an alternative definition of Hodge structure. The choice of a decomposition is equivalent to the choice of the decreasing filtration (called *Hodge filtration*)

$$H \supset \cdots \supset F^{p+1} \supset F^p \supset F^{p-1} \supset \cdots \supset (0)$$

defined by $F^p = \bigoplus_{i \leq p} H^{i,m-i}$. Then $H^{p,q} = F^p \cap \overline{F^q}$ yields the original decomposition.
Conversely, any decreasing filtration verifying the condition $F^p \oplus \overline{F^{m-p+1}} \cong H$ $\forall p \in \mathbb{Z}$ is the Hodge filtration of some Hodge structure of weight m.

So there is an equivalence between Hodge structures and Hodge filtrations. A morphism af Hodge structures $\varphi : H \longrightarrow H'$ of type (r, r) is a linear map preserving the lattice and such that $\varphi(F^p) \subset F'^{p+r}$. Actually, a morphism of Hodge structures *strictly* preserves the filtration F, i.e.,

$$\varphi(F^p) = F'^{p+r} \cap \mathrm{Im}(\varphi)\ \forall p.$$

1.2 Mixed Hodge structures

We introduce now more formal objects of Hodge theory, namely the mixed Hodge structures. We follow here the formalism introduced by Deligne in [HII], [HIII]. The Deligne splittings play a key role as they allow to prove strictness of morphisms of mixed Hodge structures and to show that the category of mixed Hodge structures is abelian.

Definition 1.5. Let $H_{\mathbb{R}}$ be a real vector space of finite dimension, and $H = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.
A *mixed Hodge structure* on H consists of:

- a given lattice $H_{\mathbb{Z}} \subset H$;

- an increasing filtration

$$(0) \subset \cdots \subset W_{m-1} \subset W_m \subset W_m \subset \cdots \subset H,$$

rationally defined (i.e., it is the extension of a filtration on $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes \mathbb{Q}$), called *weight filtration*;

- a decreasing filtration

$$H \supset \cdots \supset F^{p+1} \supset F^p \supset F^{p-1} \supset \cdots \supset (0)$$

  called *Hodge filtration*;

such that the two filtrations verify the condition
(*) $\forall m$ the filtration induced by $\{F^p\}$ on the graded $\mathrm{Gr}_m^W = W_m/W_{m-1}$ defines a Hodge structure on H of weight m.

The general element of the induced filtration is $F^p \,\mathrm{Gr}_m^W = (W_m \cap F^p)/W_{m-1}$.
An easy example of mixed Hodge structure is the direct sum of pure Hodge structures.

Definition 1.6. $\{H, W_m, F^p\}$, $\{H, W'_m, F'^p\}$ mixed Hodge structures.
A $\mathbb{C}$-linear map $\varphi : H \longrightarrow H'$ is a *morphism of mixed Hodge structures* if $\varphi(H_{\mathbb{Z}})) \subset H'_{\mathbb{Z}}$, $\varphi(W_m) \subset W'_{m+2r}$, $\varphi(F^p) \subset F'^{p+r}$.

A morphism of mixed Hodge structures induces a mapping

$$\mathrm{Gr}_m(\varphi) : \mathrm{Gr}_m^W \longrightarrow \mathrm{Gr}_{m+2r}^{W'}$$

which is a morphism of Hodge structures.

Lemma 1.7 ([HII]). *A morphism of mixed Hodge structures is strictly compatible with both the weight filtration and the Hodge filtration.*

Sketch of proof. There exists a common splitting $\{I^{p,q}\}$ of Hodge and weight filtration, such that

1. $F^p = \bigoplus_{q,r \geq p} I^{r,q}$;
2. $W_m = \bigoplus_{p+q \leq m} I^{p,q}$;
3. $I^{p,q} = \overline{I^{q,p}} \mod \bigoplus_{\substack{r \leq p-1 \\ s \leq q-1}} I^{r,s}$,

and this splitting is unique. This is a theorem (Deligne, Cattani-Kaplan, 1982). If $I^{p,q} = \overline{I^{q,p}}$, then H splits over $\mathbb{R}$, i.e., it is the direct sum of Hodge structures. Deligne proves Lemma 1.7 by giving the formula for $I^{p,q}$, but he does not show its uniqueness. This definition satisfies the first two conditions and $\varphi(I^{p,q}) \subset I'^{p,q}$ (functoriality) obviously. □

An important property we will use in the sequel is the following.

Theorem 1.8 ([HII]). *The category of mixed Hodge structures is abelian.*

Proof. See [HII]. The proof is accomplished by constructing a mixed Hodge structure on kernels and cokernels of morphisms of mixed Hodge structures, via Deligne splittings. □

2 Varieties with normal crossings

Let V be a compact complex analytic space of dimension n, verifying

$$\forall x \in V,\ \exists U, x \in U \underset{\text{open}}{\subset} V,\ \exists k, 1 \leq k \leq n,$$
$$U \cong \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : z_1\, z_2 \cdots z_k = 0, |z_i| < \epsilon, i = 1, \ldots, n+1\} \quad (*)$$

and $V = D_1 \cup \cdots \cup D_N$, where the D_i are compact Kähler manifolds intersecting transversally as (*) (normal crossing condition). Such a complex space is called a *variety with normal crossings*.

There exists a functorial mixed Hodge structure on the cohomology groups of a variety with normal crossings. In this section we follow the point of view of the article of Griffiths and Schmid [GS], which stays as close as possible to classical Hodge theory. There is a natural way to associate to a variety with normal crossings a double complex endowed with a Hodge filtration and a weight filtration. The spectral sequence associated to the weight filtration degenerates at rank 2. In other words the differentials d_r are zero for $r \geq 2$. It is proved by an explicit computation using the zig-zag principle in a double complex (see [BT]) and the principle of two types for differential forms (see [GS]). The fundamental point is that the differentials of the spectral sequence are morphisms of Hodge structures.

Theorem 2.1. *Let V a variety with normal crossings, $V = D_1 \cup \cdots \cup D_N$. Then $H^k(V, \mathbb{C})$ carries a functorial mixed Hodge structure. The weights of $H^k(V, \mathbb{C})$ vary from 0 to k.*

Remark 2.2. The claim is actually true even for rational coefficients.

In order to prove Theorem 2.1, we need to construct a filtration W on $H^k(V, \mathbb{C})$ verifying

$$(0) = W_0 \subset W_1 \subset \cdots \subset W_k = H^k(V, \mathbb{C}).$$

2.1 De Rham complex

When $V = D_1 \cup \cdots \cup D_N$, there is a natural way of decomposing V.
$\forall I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, N\}$ we set $D_I = D_{i_1} \cap \cdots \cap D_{i_q}$.
For fixed q,

$$D^{[q]} = \bigsqcup_{1 \leq i_1 < \cdots < i_q \leq N} D_{i_1, \ldots, i_q}.$$

For instance, let $V = D_1 \cup D_2 \cup D_3$, where D_1, D_2, D_3 are three planes meeting in a point. Then:
$D^{[1]} = D_1 \sqcup D_2 \sqcup D_3$ is the union of three *disjoint* planes,
$D^{[2]} = D_{12} \sqcup D_{13} \sqcup D_{23}$ is the disjoint union of the three lines of intersection between pairs of planes;
$D^{[3]} = D_1 \cap D_2 \cap D_3 = \{*\}$ is the intersection point.

Let us denote by $A^\bullet(D^{[q]})$ the De Rham complex of $\mathcal{C}^\infty$-forms on $D^{[q]}$.
Let $A^{p,q} = A^p(D^{[q+1]}) \ni \varphi$, $\varphi = \sum_{|I|=q+1} \varphi_I$, $\varphi_I \in A^p(D_I)$.

There is a standard $\mathbb{C}$-linear map $\delta : A^{p,q} \longrightarrow A^{p,q+1}$,
$\delta\varphi = \sum_{|J|=q+2}(\delta\varphi)_J,\ J = \{j_1, \dots, j_{q+2}\},\ 1 \leq j_1 < \cdots < j_{q+2} \leq N$,
$(\delta\varphi)_J := \sum_{l=1,\dots,q+2}(-1)^l \varphi_{j_1,\dots,\hat{j}_l,\dots,j_{q+2}}|_{D_J}$.
The definition immediately implies that $\delta^2 = 0$ holds. We have a double complex $\{A^{\bullet,\bullet}, d, \delta\}$

$$\begin{array}{ccc} A^{p,q+1} & \xrightarrow{d} & A^{p+1,q+1} \\ \big\downarrow \delta & & \big\downarrow \delta \\ A^{p,q} & \xrightarrow{d} & A^{p+1,q} \end{array}$$

where d is the exterior derivative multiplied by the factor $(-1)^q$, so that

$$\begin{cases} \delta^2 = 0 \\ d^2 = 0 \\ d\delta + \delta d = 0 \end{cases}$$

hold.

This double complex can be associated to any variety V with normal crossings. There is a Hodge structure arising from each row of it. We can use those Hodge structures in order to give a Hodge structure to $H^{\bullet}(V, \mathbb{C})$. For this purpose we need to introduce a powerful algebraic tool of homological algebra, the spectral sequence associated to a double complex, or, more generally, the spectral sequence of a filtered complex.

2.2 Spectral sequences

Definition 2.3. A *double complex* of A-modules is a family $K = \{K^{p,q}\}_{p,q\in\mathbb{Z}}$ of A-modules, with differential operators

$$d' : K^{p,q} \longrightarrow K^{p+1,q},$$

$$d'' : K^{p,q} \longrightarrow K^{p,q+1},$$

satisfying

$$\begin{cases} {d'}^2 = 0 \\ {d''}^2 = 0 \\ d'd'' + d''d' = 0. \end{cases}$$

To a double complex we can always associate a simple complex $(s(K), d)$, defined as follows:

$$s(K)^i = \bigoplus_{p+q=i} K^{p,q},$$

$$d : s(K)^i \longrightarrow s(K)^{i+1},$$

$$d = d' + d''.$$

We can associate to K two subcomplexes $({}'K, d')$, $({}''K, d'')$, where

$${}'K^n = \bigoplus_q K^{n,q},\ {}''K^n = \bigoplus_p K^{p,n}.$$

There is a canonical identification

$$H^n('K, d') = \bigoplus_q H^n(K^{\bullet,q}, d')$$

and analogously

$$H^n(''K, d'') = \bigoplus_p (K^{p,\bullet}, d'').$$

Let us consider $(K^{\bullet,q}, d')$. Then we can consider also

$$d'' : K^{p,q} \longrightarrow K^{p,q+1}.$$

This is *not* a map of complexes, because there is no compatibility with d'. But it gives a well-defined map at the cohomology level,

$$d''_* : H^p(K^{\bullet,q}, d') \longrightarrow H^{p+1}(K^{\bullet,q}, d'),$$

satisfying $(d''_*)^2 = 0$. This proves that $(H^p(K^{\bullet,q}, d'), d''_*)$ is a differential complex. We want to compute its cohomology, $H^q((H^p(K^{\bullet,q}, d'), d''_*)$.

An analogous construction can be done starting from $(K^{p,\bullet}, d'')$ and d'. We can construct $H^p(H^q(K^{p,\bullet}, d''), d'_*)$. Our aim is to investigate the relation between them. For what follows, our references are Deligne ([HII],[HIII], see also the appendix of [PS]) and Godement [G]. Take care that the notations of Deligne and Godement are different. We will follow those of Deligne.

Let $(K^\bullet, d)$ be a complex of A-modules, equipped with a decreasing filtration $\{F^p(K^\bullet)\}_{p\in\mathbb{Z}}$, such that $F^{-\infty} := \bigcup_p F^p(K^\bullet) = K^\bullet$, $F^\infty := \bigcap_p F^p(K^\bullet) = (0)$. This means we have a decreasing family of subcomplexes $K^\bullet \supset \cdots \supset F^p(K^\bullet) \supset F^{p+1}(K^\bullet) \supset \cdots \supset (0)$.

For later use, we will suppose this to be a *biregular* filtration, i.e., that $\{F^i(K^p)\}_{i\in\mathbb{Z}}$ is finite for all $p \in \mathbb{Z}$.

Definition 2.4. The spectral sequence associated to the filtration $F^p(K^\bullet)$ is defined by

$$Z_r^{p,q} = \ker(d : F^pK^{p+q} \to K^{p+q+1}/F^{p+r}K^{p+q+1})$$
$$B_r^{p,q} = F^{p+1}K^{p+q} + d(F^{p-r+1}K^{p+q-1})$$
$$E_r^{p,q} = Z_r^{p,q}/(Z_r^{p,q} \cap B_r^{p,q}).$$

Remark 2.5. We are working with A-modules but of course the definition of spectral sequence makes sense in any abelian category.

The differential d of $K^\bullet$ induces $\forall r$ homomorphisms

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

called *differentials* of the spectral sequence.

Computation shows that E_{r+1} is the cohomology of the complex

$$E_{r+1}^{p,q} = H(E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}).$$

In some sense this is a recursive way of computing cohomology!

The definition of $E_r^{p,q}$ makes sense also for $r = \infty$.

$$\begin{cases} F^{-\infty}(K^\bullet) = K^\bullet \\ F^{\infty}(K^\bullet) = (0) \end{cases}$$

$$Z_\infty^{p,q} = \ker(d : F^p K^{p+q} \to K^{p+q+1})$$
$$B_\infty^{p,q} = F^{p+1} K^{p+q} + d(K^{p+q-1})$$
$$E_\infty^{p,q} = Z_\infty^{p,q} / B_\infty^{p,q}.$$

Notice that F induces a decreasing filtration on the cohomology $H^n(K^\bullet)$ $\forall n$. Indeed, the inclusion

$$i : F^p(K^\bullet) \hookrightarrow K^\bullet$$

as a morphism of complexes yields a map

$$H^n(i) : H^n(F^p(K^\bullet)) \longrightarrow H^n(K^\bullet)$$

which is in general no longer an injection.

Then the filtration on the cohomology group is defined by

$$F^p(H^n(K^\bullet)) = \mathrm{Im}(H^n(i))$$

and computation shows that E_∞ is the graded with respect to that filtration,

$$E_\infty^{p,q} = \mathrm{Gr}_F^p H^{p+q}(K^\bullet).$$

This is usually expressed by the sentence *the spectral sequence converges* (or *abuts*) *to the filtered cohomology of* $K^\bullet$. There is a special notation for that,

$$E_r^{p,q} \Rightarrow H^{p+q}(K^\bullet).$$

As F is biregular, for every p, q there is an index $r = r(p,q)$ such that $Z_r^{p,q} = Z_{r+1}^{p,q} = \cdots = Z_\infty^{p,q}$, $B_r^{p,q} = B_{r+1}^{p,q} = \cdots = B_\infty^{p,q}$. As a consequence, also $E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_\infty^{p,q}$ holds.

Definition 2.6. If there is an integer r such that for all p, q $E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_\infty^{p,q}$ then the spectral sequence $\{E_\bullet^{p,q}\}$ is said to *degenerate at* r.

Notice that the spectral sequence degenerates ot r if and only if $d_s = 0$ $\forall s \geq r$.

As an example we will calculate now the first terms E_0, E_1, E_2 of the spectral sequence.

$$E_0^{p,q} = Z_0^{p,q} / B_0^{p,q} \cap Z_0^{p,q}$$
$$Z_0^{p,q} = \ker(d : F^p K^{p+q} \longrightarrow K^{p+q+1} / F^p K^{p+q+1}) = F^p K^{p+q}$$

because d is compatible with the filtration.

$$B_0^{p,q} = F^{p+1} K^{p+q} + d(F^{p+1} K^{p+q-1}) = F^{p+1} K^{p+q}$$

So

$$E_0^{p,q} = F^p K^{p+q}/F^{p+1}K^{p+q} = \operatorname{Gr}_F^p K^{p+q}.$$

$d_0 : E_0^{p,q} \longrightarrow E_0^{p,q+1}$ is the map induced by $d : K^p \longrightarrow K^{p+1}$.

$$E_1^{p,q} = H^{p+q}(E_0^{p,\bullet}, d_0) = H^{p+q}(\operatorname{Gr}_F^p K^\bullet)$$

$$\begin{array}{ccccccc} E_2^{p,q} = & H & (E_1^{p-1,q} & \xrightarrow{d_1} & E_1^{p,q} & \xrightarrow{d_1} & E_1^{p+1,q}) \\ & & || & & || & & || \\ & & H^{p+q-1}(\operatorname{Gr}_F^{p-1} K^\bullet) & & H^{p+q}(\operatorname{Gr}_F^p K^\bullet) & & H^{p+q+1}(\operatorname{Gr}_F^{p+1} K^\bullet) \end{array}$$

Now, there is the exact sequence of complexes

$$\begin{array}{ccccccc} 0 \to & F^{p+1}(K^\bullet)/F^{p+2}(K^\bullet) & \to & F^p(K^\bullet)/F^{p+2}(K^\bullet) & \to & F^p(K^\bullet)/F^{p+1}(K^\bullet) & \to 0 \\ & || & & & & || & \\ & \operatorname{Gr}_F^{p+1} K^\bullet & & & & \operatorname{Gr}_F^p K^\bullet & \end{array}$$

If we consider the induced long exact sequence in cohomology we find a connecting homomorphism

$$\cdots \longrightarrow H^{p+q}(\operatorname{Gr}_F^p(K^\bullet)) \xrightarrow{\partial} H^{p+q+1}(\operatorname{Gr}_F^{p+1}(K^\bullet)) \longrightarrow \cdots$$

and computation shows that $\partial = d_1$.

A special case of spectral sequence is the spectral sequence of a double complex $\{K^{\bullet,\bullet}, d', d''\}$. In this case we have two spectral sequences, associated to the two natural filtrations

$$'F : \ 'F^p = \bigoplus_{\substack{r,s\in\mathbb{Z} \\ r\geq p}} K^{r,s},$$

$$''F : \ ''F^q = \bigoplus_{\substack{r,s\in\mathbb{Z} \\ s\geq q}} K^{r,s}.$$

We will restrict to the case of first quadrant sequences, i.e., we will suppose $K^{p,q} = 0 \ \forall p < 0, q < 0$. Thus $'F,'' F$ induce biregular filtrations (which we will denote also by $'F,'' F$) on the simple complex $(s(K^\bullet), d)$ associated to K.

$$'F^p((s(K^\bullet)^n)) = \bigoplus_{\substack{r+s=n \\ r\geq p}} K^{r,s}$$

The complex $(s(K^\bullet),' F)$ is a filtered complex with spectral sequence $'E$ such that

$$'E_1^{p,q} = H^{p+q}(\operatorname{Gr}_{'F}^p K^\bullet) = H^q(K^{p,\bullet}, d''),$$

where $(K^{p,\bullet})^q = K^{p,q}$ and equalities

$$(\operatorname{Gr}_{'F}^p K^\bullet)^q = K^{p,q-p}, \quad \operatorname{Gr}_{'F}^p K^\bullet = K^{p,\bullet-p}$$

hold, so that $\mathrm{Gr}^p_{'F} K^\bullet$ can be considered as the "shift" $K^{p,\bullet}[-p]$.

$$'E_2^{p,q} = H^p(H^q(K^{\bullet,\bullet}, d''), d'_*)$$

Analogously the filtration $''F$ yields a spectral sequence $''E$ verifying

$$''E_2^{p,q} = H^q(H^p(K^{\bullet,\bullet}, d'), d''_*).$$

Since now we have considered only the spectral sequences associated to a filtered complez with a *decreasing filtration*. There is a natural way to associate a spectral sequence also to an *increasing filtration*.

Let $(K^\bullet, W)$ be an increasing filtration, i.e., satisfying $W_n \subset W_{n+1} \forall n \in \mathbb{Z}$. We can define in a canonical way a decreasing fitration F on $K^\bullet$ by posing $F^p = W_{-p}$, and consider the spectral sequence associated to F. Then we have

$$\mathrm{Gr}^p_F = \mathrm{Gr}^W_{-p},$$

so that

$${}_W E_1^{p,q} = H^{p+q}(\mathrm{Gr}^W_{-p}).$$

Finally let us recall here the definition of *shifted filtration*, which will be convenient for the sequel.

Definition 2.7. For any decreasing filtration F and any $n \in \mathbb{Z}$, the filtration $F[n]$ is defined by

$$(F[n])^p = F^{p+n}.$$

For any increasing filtration W and any $n \in \mathbb{Z}$, the filtration $W[n]$ is defined by

$$(W[n])_p = W_{p-n}.$$

2.3 De Rham theorem for varieties with normal crossings

We have already defined the De Rham complex $A^{p,q}$ of p-differential forms on $D^{[q+1]}$, where for any q

$$D^{[q]} = \bigsqcup_{1 \le i_1 < \cdots < i_q \le N} D_{i_1, \dots, i_q},$$

$$\forall I = \{i_1, \dots, i_q\} \subset \{1, \dots, N\}, \quad D_I = D_{i_1} \cap \cdots \cap D_{i_q}.$$

To the double complex $\{A^{\bullet,\bullet}, d, \delta\}$ we associate the simple complex $(A^\bullet, D)$, where $A = s(A^{\bullet,\bullet})$, $D = d + \delta$.

Theorem 2.8 (De Rham theorem for varieties with normal crossings).

$$H^k(A^\bullet, D) \cong H^k(V, \mathbb{C})$$

First of all, we want to consider a *sheafification* of the situation. We want to construct a sheaf $\mathcal{A}^{p,q}$ on V whose global sections verify $\Gamma(V, \mathcal{A}^{p,q}) = A^{p,q}$.

For each q there is a natural projection $\pi_q : D^{[q]} \longrightarrow V$. Let $\mathcal{A}^p_{D^{[q]}}$ denote the De Rham complex of sheaves on $D^{[q]}$; it is well-defined because $D^{[q]}$ is a smooth, compact Kähler manifold. Then the direct image of $\mathcal{A}^p_{D^{[q+1]}}$ through π_{q+1},

$$\mathcal{A}^{p,q} = (\pi_{q+1})_* \mathcal{A}^p_{D^{[q+1]}},$$

satisfies the desired condition.

If $U \subset V$ is an open set, we have

$$\Gamma(U, \mathcal{A}^{p,q}) = \bigoplus_{1 \leq i_1 < \cdots < i_{q+1} \leq N} \Gamma(U \cap D_{i_1,\ldots,i_{q+1}}, \mathcal{A}^p_{D_{i_1,\ldots,i_{q+1}}}).$$

It is easy to see that d and δ define morphisms of sheaves on $\mathcal{A}^{p,q}$.

Let us consider the simple complex of sheaves $(\mathcal{A}^\bullet = s(\mathcal{A}^{\bullet,\bullet}), D = d + \delta)$.

There is a morphism of sheaves $\mathbb{C}_V \longrightarrow \mathcal{A}^0 = \mathcal{A}^{0,0}$, given by

$$\begin{array}{ccc} \mathbb{C}_V(U) & \longrightarrow & \mathcal{A}^0(U) = \bigoplus_{j+1}^{N} A^0(U \cap D_j) \\ f & \longrightarrow & (f|_{U \cap D_j})_{1 \leq j \leq N}, \end{array}$$

for every open set $U \subset V$.

Then theorem 2.8 is a straightforward consequence of the following proposition.

Proposition 2.9.

$$0 \longrightarrow \mathbb{C}_V \longrightarrow \mathcal{A}^\bullet$$

is a soft resolution of $\mathbb{C}_V$.

Proof. $\mathbb{C}_V \longrightarrow \mathcal{A}^0$ is injective by definition. We will prove now the exactness of the sequence

$$0 \longrightarrow \mathbb{C}_V \longrightarrow \mathcal{A}^0 \longrightarrow \mathcal{A}^1.$$

Let U be a connected open subset of V, and

$$(f_j)_{1 \leq j \leq N} \in \mathcal{A}^0(U) = \bigoplus_{j+1}^{N} A^0(U \cap D_j).$$

Observe that each f_j is a constant function on $U \cap D_j$.

Suppose $D((f_j)_{1 \leq j \leq N}) = 0$. As $df_j = 0 \ \forall j + 1, \ldots, N$, we have $\delta(f) = 0$. In particular, every coordinate $(\delta f)_{i_1,i_2}$ of δf must be 0. Being

$$(\delta f)_{i_1,i_2} = -f_{i_2}|_{D_{i_1,i_2}} + f_{i_1}|_{D_{i_1,i_2}},$$

the f_j's must fit on the intersection, so that there exists $g \in \mathbb{C}_V(U)$ satisfying $g|_{U \cap D_j} = f_j$.

COnsider the double complex $\mathcal{A}^{\bullet,\bullet}$. We compute its cohomology using its spectral sequence. Take an open set $U \subset V$ such that condition (*) in the definition of variety with normal crossings holds:

$$U \cong \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : z_1 \cdots z_k = 0, |z_i| < \epsilon \ \forall i = 1, \ldots, n+1\}.$$

What we need to show is:

Claim. $H^k(\mathcal{A}^\bullet(U)) = 0 \ \forall k > 0$.

This is a sufficient condition for $0 \longrightarrow \mathbb{C}_V \longrightarrow \mathcal{A}^\bullet$ to be a resolution.

The sections of $\mathcal{A}^{p,q}$ over U are exactly the elements of the elements of the De Rham complex $\mathcal{A}^{p,q}(U)$ (U is itself a variety with normal crossings). If we look at the second spectral sequence associated of the double complex $\{A^{\bullet,\bullet}, d, \delta\}$,

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
(F_2) & & A^{0,2} & \longrightarrow & A^{1,2} & \longrightarrow & A^{2,2} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
(F_1) & & A^{0,1} & \longrightarrow & A^{1,1} & \longrightarrow & A^{2,1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
(F_0) & & A^{0,0} & \longrightarrow & A^{1,0} & \longrightarrow & A^{2,0} & \longrightarrow & \cdots
\end{array}$$

we get

$$E_1^{p,q} = H^q(A^{\bullet,p}(U), d),$$

where $A^{\bullet,p}(U), d)$ is the De Rham complex of a complex analytic space isomorphic to

$$\bigsqcup_{1 \le i_1 < \cdots < i_{p+1} \le k} \{(z_i) \in \mathbb{C}^{n+1} : z_1 \cdots z_k = 0, |z_i| < \epsilon\} \cap \{z_{i_1} = 0\} \cap \cdots \cap \{z_{i_{p+1}} = 0\}.$$

This is a disjoint uniont of contractible spaces, hence there is no cohomology in degree ≥ 1.

$$\begin{cases} E_1^{p,q} = (0) \ \forall q > 0 \\ E_1^{p,0} = \mathbb{C}^{\binom{k}{p+1}} \end{cases}$$

Having only one row, the spectral sequence degenerates at E_2:

$$E_2 \cong E_\infty = \mathrm{Gr}_F^p H(A^\bullet(U), D).$$

$E_2^{p,q} = 0$ for $q = 0$.
$E_2^{p,0}$ is the cohomology of $d_1 : E_1^{p,0} \longrightarrow E_1^{p+1,0}$.

$$E_1^{p,0} = H^0(A^{\bullet,p}(U)) =$$

$$= \bigoplus_{1 \le i_1 < \cdots < i_{p+1} \le k} H^0(\{(z_i) \in \mathbb{C}^{n+1} : z_1 \cdots z_k = 0, |z_i| < \epsilon\} \cap$$

$$\cap\{z_{i_1}=0\}\cap\ \dots\ \cap\{z_{i_{p+1}}=0\}) =$$

$$= \bigoplus_{1\le i_1<\dots<i_{p+1}\le k} \mathbb{C}_{i_1,\dots,i_{p+1}}.$$

The differential d_1 is induced by δ, so it is just an alternating sum.

$$E_1^{0,0}=\mathbb{C}^k \quad E_1^{1,0}=\mathbb{C}^{\binom{k}{2}} \quad E_1^{p,0}=\bigwedge^{p+1}(\mathbb{C}^k)$$

We can consider $E_1^{\bullet,0}$ as a *Koszul complex.*

Definition 2.10. Let V be a finite-dimensional vector space and let $\theta\in V$. Then the Koszul complex complex $K^\bullet(V,\theta)$ is defined by

$$\begin{aligned} &K^0(V,\theta)=V\\ &K^i(V,\theta)=\textstyle\bigwedge^{i+1}V\ \forall i\ge 0, \end{aligned}$$

with the differential induced by the map

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\delta^0} & V=K^0(V,\theta)\\ 1 & \longrightarrow & \theta \end{array}$$

and the relation

$$\forall\alpha\in K^i(V,\theta),\ \delta\alpha=(-1)^i\theta\wedge\alpha.$$

Lemma 2.11. *If $\theta\neq 0$, $K^\bullet(V,\theta)$ is a resolution of V.*

Idea of proof of Lemma 2.11. As the construction of the Koszul complex is coordinate-free, we can perform all computations by choosing a basis of V of the form $e_1=\theta,e_2,\dots,e_k$. Then, for a generic element $e_{i_1}\wedge\dots\wedge e_{i_j}$ of the basis of $K^j(V,\theta)$, with $1\le i_1<i_2<\dots<i_j\le k$, we have

$$\delta(e_{i_1}\wedge\dots\wedge e_{i_j})=\begin{cases} 0 & \text{if } i_1=1,\\ e_1\wedge e_{i_1}\wedge\dots\wedge e_{i_j} & \text{if } i_1>1.\end{cases}$$

This yields our claim directly. □

$E_1^{\bullet,0}$ is a Koszul complex, hence it has no cohomology in degree >0. We can thus compute the whole E_2 term of the spectral sequence:

$$E_2^{p,q}=\begin{cases} 0 & \text{for } p+q>0,\\ \mathbb{C} & \text{for } p=0,q=0.\end{cases}$$

We know that the spectral sequence degenerates at E_2. Then for each $0\le p\le k$ we have $E_\infty^{p,k-p}=E_2^{p,k-p}=0$. This means that for each p, $0\le p\le k$, the graded of the cohomology of $A^\bullet(U)$ is 0, i.e.,

$$F^pH^k(A^\bullet(U))=F^{p+1}H^k(A^\bullet(U)).$$

Then for all $k > 0$ we have

$$H^k(A^\bullet(U)) = F^0 H^k(A^\bullet(U)) = \cdots = F^{k+1} H^k(A^\bullet(U)) = 0.$$

This proves that $0 \longrightarrow \mathbb{C}_V \longrightarrow \mathcal{A}^\bullet$ is a resolution of sheaves. By a partition of unit argument this is *soft* resolution. The proof of Proposition 2.9 is then accomplished. $\square$

2.4 The mixed Hodge structure

There is a natural way to consider filtrations of the double complex $\{A^{\bullet,\bullet}, d, \delta\}$.

The weight filtration is defined as $W_m = \bigoplus_{s \geq -m} A^{r,s}$.

The Hodge filtration is as follows. Let us recall that $A^{r,s} = A^r(D^{[s+1]})$.

$A^\bullet(D^{[s+1]})$ has a Hodge filtration, so we can take

$$F^p(A^{r,s}) = F^p A^r(D^{[s+1]}).$$

We define

$$F^p A^\bullet = \bigoplus_{r,s} F^p A^{r,s}.$$

Let us consider the filtered complex $(A^\bullet, W)$, with associated spectral sequence $\{{}_W E_r^{p,q}\}$. For general spectral sequences the following hold:

$$\begin{array}{l} {}_W E_1^{p,q} = H^{p+q}(\mathrm{Gr}^W_{-p} A^\bullet); \\ \mathrm{Gr}^W_{-p} A^\bullet = A^\bullet(D^{[p+1]})[-p]; \\ {}_W E_r^{p,q} \Rightarrow H^{p+q}(A^\bullet). \end{array}$$

Together with Theorem 2.8, they imply

$${}_W E_1^{p,q} = H^q(D^{[p+1]}) \Rightarrow H^{p+q}(V, \mathbb{C}),$$

i.e., the spectral sequence of the filtered complex $\{A^\bullet, W\}$ converges to the filtered cohomology $H^{p+q}(V, \mathbb{C})$.

$H^q(D^{[p+1]})$ has a Hodge structure of weight q, induced by the filtration F. Now consider the first differential of the spectral sequence:

$$\begin{array}{cccc} d_1 : & {}_W E_1^{p,q} & \longrightarrow & {}_W E_1^{p+1,q} \\ & \| & & \| \\ & H^q(D^{[p+1]}) & & H^q(D^{[p+2]}). \end{array}$$

d_1 is induced by the differential $\delta : A^{q,p} \longrightarrow A^{q,p+1}$. As δ sends (i,j)-terms to (i,j)-terms, d_1 is a morphism of Hodge structures.

$${}_W E_2^{p,q} = H({}_W E_1^{p-1,q} \xrightarrow{d_1} {}_W E_1^{p,q} \xrightarrow{d_1} {}_W E_1^{p+1,q}).$$

The category of Hodge structures of weight q is abelian, hence ${}_W E_2^{p,q}$ has a Hodge structure of weight q. We do not need to consider any further space, because we can prove:

Lemma 2.12. *The spectral sequence $\{{}_W E_r^{p,q}\}$ degenerates at ${}_W E_2$, i.e.,*

$$ {}_W E_2^{p,q} \cong {}_W E_\infty^{p,q} \cong \operatorname{Gr}_{-p}^W H^{p+q}(V, \mathbb{C}). $$

As a consequence, $\operatorname{Gr}_{-p}^W H^{p+q}(V,\mathbb{C})$ carries a Hodge structure of weight q induced by the filtration F. Let $k = p+q$ be fixed. We have defined the shifted filtration

$$ (W[k])_p = W_{p-k}; $$

$$ \operatorname{Gr}_{-p}^W H^k(V,\mathbb{C}) = \operatorname{Gr}_q^{W[k]} H^k(V,\mathbb{C}). $$

Then $\{H^k(V,\mathbb{C}_V), \tilde{W} := W[k], F\}$ is a mixed Hodge structure. The weights vary from 0 to k:

$$ (0) = (W[k])_1 \subset (W[k])_2 \subset \cdots \subset (W[k])_k = H^k(V,\mathbb{C}). $$

Remark 2.13. It is important to realize that the weight filtration $\tilde{W} = W[k]$ of the mixed Hodge structure on $H^k(V,\mathbb{C}_V)$ is not the same W we defined before, but its shift. We have:

$$ \tilde{W}_q H^{p+q}(A^\bullet) := \operatorname{Im}(H^{p+q}((W[p+q])_q A^\bullet) \longrightarrow H^{p+q}(A^\bullet)), $$

so that

$$ \tilde{W}_q H^{p+q}(A^\bullet) = \operatorname{Im}(H^{p+q}(W_{-p}A^\bullet) \longrightarrow H^{p+q}(A^\bullet)). $$

The reason of the definition of $\tilde{W}$ is that the spectral sequence converges to the graded of $W[p+q]$ and not to that of W:

$$ E_r^{p,q} \Rightarrow \operatorname{Gr}_q^{W[p+q]} A^{p+q}(V,\mathbb{C}). $$

The proof of the degeneracy at rank 2 of the spectral sequence follows from an explicit computation of the differentials. This computation is achieved by the so-called *zig-zag principle* (see for example the appendix of the book of Bott and Tu [BT]).

2.5 The zig-zag principle for spectral sequences

Let $\{K^{p,q}, \delta, d\}$ be a double complex, with horizontal operator δ and vertical operator d,

$$ \begin{array}{ccc} K^{p,q+1} & & \\ \uparrow d & & \\ K^{p,q} & \xrightarrow{\delta} & K^{p+1,q} \end{array} $$

Then we can consider the associated simple complex $K^\bullet = s(K^{\bullet,\bullet})$ with the first filtration,

$$F^p = \bigoplus_{\substack{r,s \\ r\geq p}} K^{r,s}.$$

It is possible to give a direct description of the spectral sequence associated to $(K^\bullet, F)$ in term of the differentials d and δ.

First of all, we recall that the spectral sequence is characterized by the following property:

$$E_{r+1}^{p,q} = H(E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}).$$

As a consequence, an element $x \in K^{p,q}$ represents a cohomology class in $E_r^{p,q}$ if and only if it is a *cocycle* in $E_1^{p,q}, \dots, E_{r-1}^{p,q}$, i.e., if all differentials of order $\leq r-1$ vanish on the cohomology class of x. When it exists, we will denote the cohomology class of x in $E_r^{p,q}$ by $[x]_r$.

Let us see explicitly how this fact applies to the first terms of the spectral sequence.

$r = 1$:

$$E_1^{p,q} = H_d^q(K^{p,\bullet})$$

$$d_1 : E_1^{p,q} \longrightarrow E_1^{p+1,q}$$

Let $x \in K^{p,q}$. Then $[x]_1$ is well-defined if and only if $dx = 0$. d_1 is induced by δ, so we have $d_1[x] = [\delta x]_1$.

$r = 2$:

$$d_2 : E_2^{p,q} \longrightarrow E_2^{p+2,q-1}$$

Let $x \in K^{p,q}$. Then $[x]_2$ is well-defined if and only if the following hold:

- $dx = 0$; this implies that $[x]_1 \in E_1^{p,q}$ is defined;
- $d_1[x]_1 = 0$.

By definition, $d_1[x]_1 = [\delta x]_1$. As this is 0, there exists $y \in K^{p+1,q-1}$ satisfying $\delta x = dy$.

Now, δy represents a cohomology class in $E_2^{p+2,q-1}$. Indeed,

$$d\delta y = -\delta dy = \delta^2 x = 0,$$

$$d_1[\delta y]_1 = [\delta^2 y]_1 = [0]_1 = 0.$$

Computation shows that $d_2[x]_2 = [\delta y]_2$. Note that this result does not depend on the choice of the representatives $x \in K^{p,q}$, $y \in K^{p+1,q-1}$.

We can see all this as a zig-zag in the double complex:

$$\begin{tikzcd}
0 & & \\
x \arrow[u, "d"] \arrow[r, "\delta"] & {} & \\
 & y \arrow[u, "d"] \arrow[r, "\delta"] & {}
\end{tikzcd}$$

We can summarize all this by saying that $x \in K^{p,q}$ represents a cohomology class in $E_2^{p,q}$ if and only if x *can be extended to a zig-zag of length 2.*

$r = 3$:

$$d_3 : E_3^{p,q} \longrightarrow E_3^{p+3,q-2}$$

Let $x \in K^{p,q}$. Then $[x]_3$ is well-defined if and only if the following hold:

- $dx = 0$; this implies that $[x]_1 \in E_1^{p,q}$ is defined;
- $d_1[x]_1 = [\delta x] = 0$. Then, there exists $y_1' \in K^{p+1,q-1}$ such that $\delta x = dy_1'$;
- $d_2[x_2] = [\delta y_1']_2 = 0$. Then, $\exists y_1'' \in K^{p+1,q-1} : dy_1'' = 0$, $d_1[y_1'']_1 = [\delta y_1']_1$.

Let $y_1 = y_1' - y_1''$. $\delta y_1 \in E^{p+2,q-1}$, because

$d\delta y_1 = -\delta dy = \delta^2 = 0$;

$d_1[\delta y_1]_1 = [\delta y_1' - \delta y_1'']_1 = [\delta y_1']_1 - [\delta y_1']_1 = 0$.

As $dy_1 = dy_1' - dy_1'' = \delta x$, we have $[\delta y_1]_2 = d_2[x]_2 = 0$. Then $\exists y_2 \in K^{p+2,q-2}$ with $y_2 = \delta y_1$.

Summarizing, we are in the following situation:

$$\begin{aligned} dy_1 &= \delta x, \\ dy_2 &= \delta y_1. \end{aligned}$$

Analogously to the previous cases, we can now conclude

$$d_3[x]_3 = [\delta y_2]_3.$$

In other words, we have shown that $x \in K^{p,q}$ represents a cohomology class in $E_3^{p,q}$ if and only if it can be extended to a zig-zag of length 3.

$$\begin{tikzcd}
0 & & & \\
x \arrow[u, "d"] \arrow[r, "\delta"] & {} & & \\
 & y_1 \arrow[u, "d"] \arrow[r, "\delta"] & {} & \\
 & & y_2 \arrow[u, "d"] \arrow[r, "\delta"] & {}
\end{tikzcd}$$

General case: $x \in K^{p,q}$ represents a cohomology class $[x]_r \in E_r^{p,q}$ if and only if x can be extended to a zig-zag of length r.

$$\begin{array}{cccccccccccc}
0 &&&&&&&&&&& \\
\uparrow{\scriptstyle d} &&&&&&&&&&& \\
x & \xrightarrow{\delta} &&&&&&&&&& \\
&& \uparrow{\scriptstyle d} &&&&&&&&& \\
&& y_1 & \xrightarrow{\delta} &&&&&&&& \\
&&&& \uparrow{\scriptstyle d} &&&&&&& \\
&&&& y_2 & \xrightarrow{\delta} & \cdots &&&&& \\
&&&&&&& \uparrow{\scriptstyle d} &&&& \\
&&&&&&& y_{r-2} & \xrightarrow{\delta} &&& \\
&&&&&&&&& \uparrow{\scriptstyle d} && \\
&&&&&&&&& y_{r-1} & \xrightarrow{\delta} &
\end{array}$$

(In the diagram y_j represents a cohomology class $[y_j]_{j+1}$.)

The differential $d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-3}$ is given by the δ of the tail of the zig-zag:

$$d_r[x]_r = [\delta y_{r-1}]_r.$$

We can also deduce the zig-zag principle directly from the filtered complex $(s(K)^\bullet, D)$. Consider first $E_1^{p,q} = Z_1^{p,q}/(Z_1^{p,q} \cap B_1^{p,q})$. Then an element of $K^{p,q}$ belongs to $Z_1^{p,q}$ if and only if its vertical differential d is 0. In this case we easily find that $d_1[x]_1 = 0$. Vice versa, every element of $E_1^{p,q}$ can be represented by an element $x \in K^{p,q} \cap Z_1^{p,q}$.

Now, take into account $E_2^{p,q}$. The intersection $K^{p,q} \cap Z_2^{p,q}$ is not particularly interesting, being simply made up by elements $z \in K^{p,q}$ such that $dz = \delta z = 0$. Anyway, an element $x \in K^{p,q} \cap Z_1^{p,q}$ represents a class in $E_2^{p,q}$ if there exists $y \in K^{p+1,q} \subset B_2^{p,q}$ such that $x + y \in Z_2^{p,q}$. This is exactly the same condition as requiring the existence of a zig-zag of length 2. As before, we find that $d_2[x]_2 = [\delta y]_2$.

Analogously, an element $x \in K^{p,q}$ represents a class in $E_r^{p,q}$ if and only if

- $x \in Z_1^{p,q}$;
- $\exists y_1 \in K^{p+1,q}$ such that $x + y_1 \in Z_2^{p,q}$;
- $\exists y_2 \in K^{p+2,q}$ such that $x + y_1 + y_2 \in Z_3^{p,q}$;
- ....
- $\exists y_{r-1} \in K^{p+r-1,q}$ such that $x + y_1 + \cdots + y_{r-1} \in Z_r^{p,q}$.

This is exactly the same as requiring the existence of a zig-zag of length r.

As we already said, the zig-zag principle provides us a simple way of computing the elements and the differentials in a spectral sequence. We will apply this in the case of the cohomology of a variety with normal crossings.

Let us consider a variety with normal crossings $V = D_1 \cup \cdots \cup D_N$, with its double complex $(A^{p,q}_{,}d, \delta)$. We defined

$$W_n = \bigoplus_{\substack{r,s\in\mathbb{Z}\\ s\geq -n}} A^{r,s}.$$

This is the *second* filtration associated to the double complex $A^{\bullet,\bullet}$. In the previous computations of this section we always worked with the first filtration. For the second filtrations all results are of course the same, but we need to reverse all the order in all pairs of indices considered.

We were interested in computing the spectral sequence associated to the filtered complex $\{A^\bullet, W\}$. We remarked that ${}_WE_1^{p,q} = H^q_{DR}(D^{p+1})$, and that the fact that d_1 is a morphism of Hodge structures of weight q gives $E_2^{p,q}$ a Hodge structure of the same weight.

Proof of Lemma 2.12.

We have claimed the degeneration at ${}_WE_2$ of the spectral sequence $\{{}_WE_r^{p,q}\}$, defined on the cohomogy groups of a variety with normal crossings. First of all, we compute $d_2 : {}_WE_2^{p,q} \longrightarrow {}_WE_2^{p+2,q-1}$.

Let $x \in {}_WE_2^{p,q}$; then it is represented by some $\omega \in A^{q,p} = A^q(D^{[p+1]})$. We can extend ω to a zig-zag of length 2 in the spectral sequence.

$$\begin{tikzcd} \beta \arrow[r, "d"] & {} \\ & \omega \arrow[u, "\delta"] \arrow[r, "d"] & 0 \end{tikzcd}$$

The zig-zag principle establishes the existence of $\beta \in A^{q-1}(D^{[p+2]})$ such that $d\beta = \delta\omega$. For any such β we have $d_2[\omega]_2 = [\delta\beta]_2$.

We can assume without loss of generality that ω is a q-form of pure type (r, s) (otherwise, we can decompose it into a sum of such). By the principle of two types (see [GS]), we can choose between two possibilities for β. The differential $\delta\omega$ is an exact form of type (r, s) (because δ preserves the type). Then there exists β' of type $(r-1, s)$ such that $d\beta' = \delta\omega$ and also β'' of type $(r, s-1)$ such that $d\beta'' = \delta\omega$. This implies

$$[\delta\beta']_2 = d_2[\omega]_2 = [\delta\beta'']_2,$$

where $[\delta\beta']_2$ is of type $(r-1, s)$ in ${}_WE_2^{p+2,q-1}$ and $[\delta\beta'']_2$ is of type $(r, s-1)$ in the same space. Then, by Hodge decomposition on ${}_WE_2^{p+2,q-1}$, it must be $d_2[\omega]_2 = 0$. This proves that d_2 is the 0 morphism.

Consider now d_3. We have ${}_WE_3^{p,q} = {}_WE_2^{p,q}$, so that ${}_WE_3^{p,q}$ carries a Hodge structure of weight q induced by the differential of $A^\bullet$. Let $\omega \in A^q(D^{[p+1]})$ be a q-form that represents a cohomology class in ${}_WE_2^{p,q}$. We can assume that ω is of pure type (r, s).

By the zig-zag principle, we can extend ω to a zig-zag of length 3:

$$\begin{array}{ccccc}
\beta_2 \xrightarrow{d} & & & & \\
& \delta\Big\uparrow & & & \\
& \beta_1 \xrightarrow{d} & & & \\
& & \delta\Big\uparrow & & \\
& & \omega \xrightarrow{d} 0. & &
\end{array}$$

We may assume that β_1 is a form of pure type $(r-1, s)$. Let us start with a general β_1' such that $d\beta_1' = \delta\omega$ Then there exists β_1'' verifying

$$\begin{cases} [\delta\beta_1'']_1 = [\delta\beta_1']_1, \\ d\beta_1'' = 0. \end{cases}$$

As $[\delta\beta_1'']_1$ has type $(r-1, s)$, β_1'' can be chosen of type $(r-1, s)$. We can perform this just by taking an arbitrary β_1'' and replacing it by $\beta_1'' - \beta_1'''$, where β_1''' is a form of type $(r-1, s)$ satisfying

$$\begin{cases} [\beta_1''']_1 = [\beta_1'']_1 \\ d\beta_1''' = 0. \end{cases}$$

Then $\beta_1 = \beta_1' - \beta_1''$ is a form of type $(r-1, s)$, which satisfies the conditions for belonging to the zig-zag of ω.

We are now in the same situation as before. By the principle of two types, there are two possibilities for β_2:

$$\begin{array}{ll} \exists\beta_2' \text{ of type } (r-2, s) & \text{such that } d\beta_2' = \delta\beta_1, \\ \exists\beta_2'' \text{ of type } (r-1, s-1) & \text{such that } d\beta_2'' = \delta\beta_1. \end{array}$$

As a consequence, $d_3[\omega]_3$ must be of both type $(r-2, s)$ and type $(r-1, s-1)$ in ${}_WE_3^{p,q}$. We have then $d_3[\omega]_3 = 0$.

We can prove analogously that $d_r = 0$ for all $r \geq 2$. The basic reason is that d_r is a morphism of Hodge structures, but any non-trivial morphism yields a contradiction with the principle of two types on $(A^\bullet(D^{[p+1]}), d)$. $\square$

3 Smooth quasi-projective varieties

In *Théorie de Hodge II* [HII], Deligne gives the construction of a functorial mixed Hodge structure on the cohomology of smooth quasi-projective algebraic varieties which coincides with the classical Hodge structure when the variety is smooth and projective.

Theorem 3.1 (Deligne). *Let U be a smooth, quasi-projective algebraic variety over $\mathbb{C}$. Then the integral cohomology $H^\bullet(U, \mathbb{Z})$ has a functorial mixed Hodge structure. The weights on $H^k(U, \mathbb{Q})$ vary from k to $2k$.*
The Hodge numbers $h^{p,q}(H^k(U, \mathbb{C}) = 0$ unless $0 \leq p, q \leq k$, $k \leq p + q \leq 2k$.

We recall here the definition of the *Hodge numbers* for a mixed Hodge structure:

$$h^{p,q}(H^k(U,\mathbb{C}) = \dim_{\mathbb{C}}(\mathrm{Gr}^W_{p+q} H^k(U,\mathbb{C}))^{p,q}.$$

Actually in this section we do not prove this theorem yet but only collect data concerning the cohomology in a nice way. Precisely we construct an *integral mixed Hodge complex of sheaves* according to the terminology of Deligne [HIII]. Then the theorem will be a consequence of the general Hodge theory of Deligne which we explain in Section 5.

The construction starts with the following observation: By Nagata's theorem on compactification of algebraic variety and Hironaka's theorem on resolution of singularities, for every quasi-projective complex algebraic variety U there exists a smooth compactification X such that

- X is a projective variety;
- $\exists D \subset X$, normal crossing divisor with smooth projective components, such that $X - D$ is isomorphic to U.

The mixed Hodge structure on U is constructed via the holomorphic logarithmic De Rham complex (see [D]), to which we devote the next section. Deligne shows that the mixed Hodge structure actually does not depend on the choice of a compactification.

3.1 The logarithmic complex

Let X be a complex manifold, and $D \subset X$ a normal crossing divisor. Then we will denote $X^* = X - D$, with the natural inclusion $j: X^* \hookrightarrow X$.

Definition 3.2. Let X be a complex manifold and $D \subset X$ a divisor with normal crossings. Then the sheaf of *holomorphic differential forms with logarithmic poles along* D is the sheaf $\Omega^\bullet_X(\log D)$ on X defined by the condition that a section φ of $j_*\Omega^\bullet_{X^*}$ is a section of $\Omega^\bullet_X(\log D)$ if and only if for any local equation f defining D then both $f\varphi$ and $f\mathrm{d}\varphi$ are local sections of $\Omega^\bullet_X$.

We have $\Omega^p_X(\log D) \subset \Omega^{p+1}_X(\log D)$, so that d gives $\Omega^\bullet_X(\log D)$ the structure of a differential complex.

Definition 3.3. The differential complex $(\Omega^\bullet_X(\log D), \mathrm{d})$ is called the *holomorphic logarithmic De Rham complex.*

By definition, we have

$$\Omega^\bullet_X \subset \Omega^\bullet_X(\log D) \subset j_*\Omega^\bullet_{X^*}.$$

Proposition 3.4. *$\Omega^1_X(\log D)$ is a locally free sheaf of $\mathcal{O}_X$-modules, and*

$$\Omega^p_X(\log D) = \bigwedge^p \Omega^1_X(\log D).$$

Proof. Consider a coordinate open set $U \subset X$ with holomorphic coordinates $(z_1, \dots, z_n)$ such that

$$U \cap D = \{z_1 \cdots z_k = 0\}.$$

Let $\Omega^1_X(U)\left\{\frac{\mathrm{d}z_1}{z_1}, \frac{\mathrm{d}z_2}{z_2}, \dots, \frac{\mathrm{d}z_k}{z_k}\right\}$ be the free $\mathcal{O}_X$-module with basis

$$\frac{\mathrm{d}z_1}{z_1}, \dots, \frac{\mathrm{d}z_k}{z_k}, \mathrm{d}z_1, \dots, \mathrm{d}z_n.$$

Let $\Omega^p_X(U)\left\{\frac{\mathrm{d}z_1}{z_1}, \frac{\mathrm{d}z_2}{z_2}, \dots, \frac{\mathrm{d}z_k}{z_k}\right\} = \bigwedge^p \Omega^1_X(U)\left\{\frac{\mathrm{d}z_1}{z_1}, \frac{\mathrm{d}z_2}{z_2}, \dots, \frac{\mathrm{d}z_k}{z_k}\right\}$.
Note that for every p, $\Omega^p_X(U)\left\{\frac{\mathrm{d}z_1}{z_1}, \frac{\mathrm{d}z_2}{z_2}, \dots, \frac{\mathrm{d}z_k}{z_k}\right\}$ can be naturally identified with a submodule of $j_*\Omega^p_{X*}(U)$.

Now, we need only to prove that

$$\Omega^p_X(U)\left\{\frac{\mathrm{d}z_1}{z_1}, \frac{\mathrm{d}z_2}{z_2}, \dots, \frac{\mathrm{d}z_k}{z_k}\right\} = \Gamma(U, \Omega^p_X(\log D)).$$

Suppose $\varphi \in \Omega^p_X(U)\left\{\frac{\mathrm{d}z_1}{z_1}, \frac{\mathrm{d}z_2}{z_2}, \dots, \frac{\mathrm{d}z_k}{z_k}\right\}$. Without loss of generality, we can assume

$$\varphi = \omega \wedge \frac{\mathrm{d}z_{i_1}}{z_{i_1}} \wedge \frac{\mathrm{d}z_{i_2}}{z_{i2}} \wedge \cdots \wedge \frac{\mathrm{d}z_{i_q}}{z_{i_q}},$$

with $\omega \in \Omega^{p-q}_X$, $1 \le q \le k$.

Then $z_1 \cdots z_k \varphi \in \Omega^p_X(U)$, and

$$\mathrm{d}\varphi = \mathrm{d}\omega \wedge \frac{\mathrm{d}z_{i_1}}{z_{i_1}} \wedge \frac{\mathrm{d}z_{i_2}}{z_{i2}} \wedge \cdots \wedge \frac{\mathrm{d}z_{i_q}}{z_{i_q}},$$

so that $z_1 \cdots z_k \mathrm{d}\varphi \in \Omega^{p+1}_X(U)$. This proves $\varphi \in \Omega^p_X(\log D)$.

In order to establish the other inclusion, we restrict ourselves to the case $k = 1$. The proof for a general k is essentially the same (see [GH, p. 450]).

Let $\alpha \in \Gamma(U, \Omega^p_X(\log D))$. Then $z_1\alpha \in \Omega^p_X$. This means that we can write $z_1\alpha = \beta \wedge \mathrm{d}z_1 + \gamma$, where β and γ do not involve $\mathrm{d}z_1$. We have

$$\alpha = \beta \wedge \frac{\mathrm{d}z_1}{z_1} + \frac{\gamma}{z_1},$$

$$\mathrm{d}\alpha = \mathrm{d}\beta \wedge \frac{\mathrm{d}z_1}{z_1} - \frac{1}{{z_1}^2}\mathrm{d}z_1 \wedge \gamma + \frac{1}{z_1}\mathrm{d}\gamma.$$

By definition we have $z_1\mathrm{d}\alpha \in \Omega^{p+1}_X(U)$, so that $\frac{\gamma}{z_1} \in \Omega^p_X(U)$ and $\alpha \in \Omega^p_X(U)\left\{\frac{\mathrm{d}z_1}{z_1}\right\}$. □

3.2 The weight filtration of the logarithmic complex

We can construct now the *weight filtration* of the logarithmic complex. The definition is really natural, being based on the order of poles of the forms:

$$W_m\Omega_X^p(\log D) = \begin{cases} \Omega_X^m(\log D) \wedge \Omega_X^{p-m} & \text{for } m < p, \\ \Omega_X^p(\log D) & \text{for } m \geq p. \end{cases}$$

In a neighbourhood of a point $x \in D$, we can choose holomorphic coordinates $z_1, \ldots, z_n$, such that D is locally defined by $z_1 \cdots z_k = 0$. We have then

$$(W_m\Omega_X^p(\log D))_x = \sum_{i_1 < \cdots < i_m} \Omega_{X,x}^{p-m} \wedge \frac{\mathrm{d}z_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}z_{i_m}}{z_{i_m}}.$$

The weight filtration is an increasing filtration compatible with the structure of differential complex:

$$\begin{gathered} W_m \subset W_{m+1}, \\ \mathrm{d}W_m\Omega_X^p(\log D) \subset W_m\Omega_X^{p+1}(\log D), \\ (W_m\Omega_X^\bullet(\log D), \mathrm{d}) \subset (\Omega_X^\bullet(\log D), \mathrm{d}). \end{gathered}$$

3.3 The Poincaré residue map

Next topic is the *Poincaré residue map*. In order to define it, we need to use a special class of holomorphic systems of coordinates. First of all, we consider a decomposition of the divisor $D = D_1 \cup \cdots \cup D_N$. This decomposition is unique but for the order of the smooth component D_j. From now on we consider this order as fixed, i.e., we choose a bijection between the set of smooth components of D and the set of indices $1, \ldots, N$.

By the definition of normal crossing divisor, we have that every point of D admit holomorphic coordinates $z_1, \ldots, z_n$ in an open neibourhood $U \subset X$, satisfying the condition:

$$D \text{ is given by the local equation } z_1 \cdots z_k = 0 \text{ for some } k \geq 1. \tag{*}$$

For such coordinates, we have $\{z_i = 0\} \subset D_{n_i}$ for every i, $1 \leq i \leq k$. We want the order of the z_i's to agree with that of the D_j's. Formally, this additional condition is:

$$n_1 < n_2 < \cdots < n_k \tag{**}$$

This is not a restrictive condition, because it only require us to reorder $z_1, \ldots, z_k$ in a certain way.

Definition 3.5. Suppose $U \subset X$ is an open set with coordinates $z_1, \ldots, z_n$ satisfying the conditions above. Let $m \leq k$, fix a m-uple $I = (i_1, \ldots, i_m)$, with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$. We define

$$\begin{array}{rccc} \rho_0^I : & \Gamma(U, \Omega_X^p) & \longrightarrow & \Gamma(U, \mathrm{Gr}_m^W(\Omega_X^{p+m}(\log D))) \\ & \alpha & \longrightarrow & \frac{\mathrm{d}z_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}z_{i_m}}{z_{i_m}} \wedge \alpha. \end{array}$$

Lemma 3.6. *The definition of ρ_0^I does not depend on the choice of the coordinates (provided they satisfy conditions (*) and (**)).*

Proof. Let $f : U \longrightarrow U'$ be an isomorphism. Let $z'_i = f_i(z_1, \dots, z_n)$ $\forall 1 \leq i \leq n$. Assume that the z_i's satisfy conditions (*) and (**), and that

$$f(U \cap \{z_i = 0\}) = U' \cap f(\{z_i = 0\}) = \{z'_i = 0\}.$$

What we need to show is $f^*\rho'_0(\alpha) = \rho_0(f^*\alpha)$, where ρ'_0 denotes the map ρ_0 in the coordinates z'_i. We consider the multiindex I as fixed, so we omit it. We have:

$$\rho'_0(\alpha) = \frac{\mathrm{d}z'_{i_1}}{z'_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}z'_{i_m}}{z'_{i_m}} \wedge \alpha,$$

$$f^*\rho'_0(\alpha) = f^*\left(\frac{\mathrm{d}z'_{i_1}}{z'_{i_1}}\right) \wedge \cdots \wedge f^*\left(\frac{\mathrm{d}z'_{i_m}}{z'_{i_m}}\right) \wedge f^*(\alpha),$$

where $f^*\left(\frac{\mathrm{d}z'_j}{z'_j}\right) = \frac{\mathrm{d}f_j}{f_j}$.

We have $\{f_i(z_1, \dots, z_n) = 0\} = U \cap D_{n_i} = \{z_i = 0\}$. As D_i is a divisor, $\frac{f_i(z_1,\dots,z_n)}{z_i}$ must be a holomorphic function on U with holomorphic inverse. Taking the logarithmic differential of it we get

$$\frac{\mathrm{d}f_i}{f_i} = \frac{\mathrm{d}z_i}{z_i} + \beta_i,$$

where β_i is a holomorphic 1-form on U. Then

$$f^*\rho'_0(\alpha) = \frac{\mathrm{d}z_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}z_{i_m}}{z_{i_m}} \wedge f^*(\alpha) \pmod{W_{m-1}\Omega_X^{p+m}(\log D)},$$

hence they coincide as elements of $\Gamma(U, \mathrm{Gr}_m^W \Omega_X^{p+m}(\log D))$.

□

Remark 3.7. The condition (**) is important because the interchange of a pair of the chosen coordinates changes the sign of ρ_0. This means that ρ_0 depends only on the orientation of the basis chosen.

Corollary 3.8. ρ_0^I *induces a map of sheaves*

$$\rho_0^I : \Omega_X^p \longrightarrow \mathrm{Gr}_m^W \Omega_X^{p+m}(\log D).$$

As before, we consider $D_I = D_{i_1} \cap \cdots \cap D_{i_m}$. Let we denote by $a_I : D_I \hookrightarrow X$ the natural inclusion. a_I defines a surjective map of sheaves

$$\Omega_X^p \twoheadrightarrow (a_I)_* \Omega_{D_I}^p,$$

induced by the natural projection of local holomorphic coordinates $U \cap X \twoheadrightarrow D_I$.

This leads to the following diagram:

$$\begin{tikzcd}
\Omega^p_X \arrow[rr, "\rho_0^I"] \arrow[dr] & & \mathrm{Gr}^W_m \Omega^{p+m}_X(\log D). \\
& (a_I)_*\Omega^p_{D_I} \arrow[ur, dotted] &
\end{tikzcd}$$

Let $\alpha \in \Omega^p_X$ such that $(a_I)^*\alpha = \alpha_{D_I} = 0$. We have that $\alpha_{D_I} = 0$ if and only if, in local coordinates satisfying (*), α is a linear combination of elements of the form $z_{i_\lambda}\beta$ and $\mathrm{d}z_{i_\lambda} \wedge \beta$ for $1 \leq \lambda \leq k$. But

$$\rho^I_0(z_{i_\lambda})\beta) = \frac{\mathrm{d}z_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}z_{i_m}}{z_{i_m}} \wedge z_{i_\lambda \beta} \in W_{m-1}\Omega^{p+m}_X(\log D)$$

is 0 in the graded, and

$$\rho^I_0(\mathrm{d}z_{i_\lambda} \wedge \beta) = 0.$$

In this way we get a map of sheaves

$$\rho^I_1 : (a_I)_*\Omega^p_{D_I} \longrightarrow \mathrm{Gr}^W_m \Omega^{p+m}_X(\log D).$$

If we consider $D^{[m]} = \displaystyle\bigsqcup_{\substack{I=(i_1,\ldots,i_m)\\ i_1<\cdots<i_m}} D_I$ and the map

$$a_m : D^{[m]} \longrightarrow X,$$

$$a_m = \bigsqcup_{\substack{I=(i_1,\ldots,i_m)\\ i_1<\cdots<i_m}} a_I,$$

the sum of the morphisms ρ^I_1 defines a map

$$\rho : (a_m)_*\Omega^p_{D^{[m]}} \longrightarrow \mathrm{Gr}^W_m \Omega^{p+m}_X(\log D).$$

The map ρ is a chain map, and even a map of complexes of sheaves if we multiply it by a factor $(-1)^p$. In fact, we can consider ρ as a map of complexes without need of any change, if we accept the convention that for a complex $(K^\bullet, \mathrm{d})$ the shifted complex is defined by $K^\bullet[m] = (K^{\bullet+m}, (-1)^m\mathrm{d})$.

Proposition 3.9.

$$\rho : (a_m)_*\Omega^\bullet_{D^{[m]}}[-m] \longrightarrow \mathrm{Gr}^W_m \Omega^\bullet_X(\log D)$$

is an isomorphism of complexes of sheaves.

We omit here the proof of Proposition 3.9. It consists of the explicit construction of the inverse of ρ. The proof that this is a well-defined map of complexes follows from the same arguments used for ρ.

Definition 3.10. The inverse of ρ is called *Poincaré residue map.*
It is given $\forall m$ by the morphism of sheaves

$$\mathrm{Res}_m : \mathrm{Gr}^W_m \,\Omega^{p+m}_X(\log D) \xrightarrow{\sim} (a_m)_* \Omega^p_{D^{[m]}}$$

whose expression in local coordinates $z_1, \dots, z_n$ satisfying conditions (*) and (**) is

$$\sum_{i_1 < \cdots < i_m} \frac{\mathrm{d}z_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}z_{i_m}}{z_{i_m}} \wedge \alpha_{(i_1,\dots,i_m)} \longrightarrow \sum_{i_1 < \cdots < i_m} \alpha_{(i_1,\dots,i_m)}|_{D_{i_1} \cap \cdots \cap D_{i_m}}.$$

3.4 Hypercohomology of the logarithmic complex

From the holomorphic Poincaré lemma we know that

$$0 \longrightarrow \mathbb{C}_{D[m]} \longrightarrow \Omega^\bullet_{D^{[m]}}$$

is a resolution. The morphism $a_m : D^{[m]} \longrightarrow X$ is finite and proper, hence

$$0 \longrightarrow (a_m)_* \mathbb{C}_{D^{[m]}} \longrightarrow (a_m)_* \Omega^\bullet_{D^{[m]}}$$

is again a resolution.

This gives us a *quasi-isomorphism*

$$(a_m)_* \mathbb{C}_{D^{[m]}}[0] \xrightarrow{\text{q. is.}} (a_m)_* \Omega^\bullet_{D^{[m]}},$$

where $\mathbb{C}_{D^{[m]}}[0]$ denotes the complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow (a_m)_* \mathbb{C}_{D^{[m]}} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots.$$

We recall here that, by definition (see for example [PS]), a quasi-isomorphism is a map of complexes such that the induced map on cohomology is an isomorphism.

This allows us to compute the cohomology sheaf of $\mathrm{Gr}^W_m \,\Omega^\bullet_X(\log D)$:

$$\mathcal{H}^i \,\mathrm{Gr}^W_m \,\Omega^\bullet_X(\log D) \cong \mathcal{H}^i((a_m)_* \mathbb{C}_{D^{[m]}}[-m])$$

so that

$$\mathcal{H}^i(\mathrm{Gr}^W_m \,\Omega^\bullet_X(\log D)) = \begin{cases} (a_m)_* \mathbb{C}_{D^{[m]}} & \text{for } i = m, \\ 0 & \text{for } i \neq m. \end{cases} \tag{†}$$

As a consequence, we have

$$\mathcal{H}^i(W_m \Omega^\bullet_X(\log D)) = \begin{cases} (a_m)_* \mathbb{C}_{D^{[m]}} & \text{for } i \leq m, \\ 0 & \text{for } i > m. \end{cases}$$

We can prove it by induction on m.

For $m = 0$, we have $\operatorname{Gr}_0^W \Omega_X^\bullet(\log D) = W_0\Omega_X^\bullet(\log D) = \Omega_X^\bullet$, and (by convention) $D^{[0]} = X$. So the claim is equivalent to (†).

For $m > 0$, the exact sequence

$$0 \longrightarrow W_{m-1}\Omega_X^\bullet(\log D) \longrightarrow W_m\Omega_X^\bullet(\log D) \longrightarrow \operatorname{Gr}_m^W \Omega_X^\bullet(\log D) \longrightarrow 0,$$

induces a long exact sequence in cohomology:

$$\mathcal{H}^{i-1}(\operatorname{Gr}_m^W \Omega_X^\bullet(\log D)) \longrightarrow \mathcal{H}^i(W_{m-1}\Omega_X^\bullet(\log D)) \longrightarrow \mathcal{H}^i(W_m\Omega_X^\bullet(\log D)) \longrightarrow$$

$$\longrightarrow \mathcal{H}^i(\operatorname{Gr}_m^W \Omega_X^\bullet(\log D)) \longrightarrow \mathcal{H}^{i+1}(W_{m-1}\Omega_X^\bullet(\log D)),$$

which gives us the inductive step $(m-1) \Rightarrow (m)$.

In particular, when $m \geq \dim_{\mathbb{C}} X$, we have $W_m\Omega_X^\bullet(\log D) = \Omega_X^\bullet(\log D)$, and

$$\mathcal{H}^i(\Omega_X^\bullet(\log D)) \cong (a_i)_*\mathbb{C}_{D^{[i]}} \ \forall i \geq 0.$$

To proceed further we need to use the notion of *hypercohomology of a complex of sheaves*. We sketch here its definition below. For a more complete treatment of this subject, see [G].

Definition 3.11. Let $\mathcal{F}^\bullet$ a complex of sheaves on the topological space X. Then to $\mathcal{F}^\bullet$ we can associate in a canonical way a flabby resolution (i. e., a resolution which is a complex of flabby sheaves), called its *Godement resolution*,

$$\mathcal{F}^\bullet \xrightarrow{\sim} \mathcal{C}^\bullet_{\mathrm{Gdm}}(\mathcal{F})^\bullet.$$

We define then the *hypercohomology* of $\mathcal{F}^\bullet$ on X to be

$$\mathbb{H}^i(X, \mathcal{F}^\bullet) := H^i(\Gamma(X, \mathcal{C}^\bullet_{\mathrm{Gdm}}(\mathcal{F}^\bullet))).$$

An important property of hypercohomology is that if two complexes of sheaves are quasi-isomorphic then they have the same hypercohomology.

Proposition 3.12. *Let us consider the Godement resolution*

$$\Omega^\bullet_{X^*} \xrightarrow{\sim} \mathcal{C}^\bullet_{\mathrm{Gdm}}(\Omega^\bullet_{X^*}).$$

Then

$$j_*\Omega^\bullet_{X^*} \xrightarrow{\text{q. is.}} j_*\mathcal{C}^\bullet_{\mathrm{Gdm}}(\Omega^\bullet_{X^*}).$$

Proof. Being flabby, $\mathcal{C}^\bullet_{\mathrm{Gdm}}(\Omega^\bullet_{X^*})$ is j_*-acyclic, i.e., $\mathcal{R}^i j_*\mathcal{C}^p_{\mathrm{Gdm}}(\Omega^\bullet_{X^*}) = 0 \ \forall i > 0$. Then it suffices to prove that also $\Omega^\bullet_{X^*}$ is j_*-acyclic. The key remark is that each point on X has a fundamental system of open neighbourhoods by Stein open sets.

Indeed, j_* does nothing in a neighbourhood of a point on X^*. If we consider a point on D, we can choose local coordinates $z_1, \ldots, z_n$ such that condition (*) holds, and we have that a open neighbourhood U has the form

$$X \supset U \cong \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| < \epsilon\},$$

and

$$D \cap U \cong \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1 \cdots z_k = 0, |z_i| < \epsilon\}.$$

Now,

$$(\mathcal{R}^i j_* \Omega^p_{X^*})_x = \varinjlim_U \Gamma(U, \mathcal{R}^i j_* \Omega^p_{X^*}) = \varinjlim_U H^i(U^*, \Omega^p_{X^*}).$$

$U^* = U - D$ is a Stein manifold (a Stein manifold is the complex analogue of an affine variety), as can be seen from the embedding

$$\begin{array}{ccc} U^* & \hookrightarrow & U \times \mathbb{C} \\ z & \to & (z, \frac{1}{z_1 \cdots z_k}). \end{array}$$

We can then apply to U^* Cartan's theorem B and conclude that

$$H^i(U^*, \Omega^p_{X^*}) = 0 \ \forall i > 0.$$

This implies the claim. □

Theorem 3.13. *The inclusion $\Omega^\bullet_X(\log D) \hookrightarrow j_* \Omega^\bullet_{X^*}$ is a quasi-isomorphism.*

Proof. We recall from the proof of Proposition 3.12 that $\Omega^\bullet_{X^*}$ is j_*-acyclic. Hence we can use the resolution

$$0 \longrightarrow \mathbb{C}_{X^*} \longrightarrow \Omega^\bullet_{X^*}$$

to compute the direct image sheaf

$$\mathcal{R}^i j_* \mathbb{C}_{X^*} = \mathcal{H}^i(j_* \Omega^\bullet_{X^*}).$$

Now, what we want to prove is local in nature. Therefore it suffices to show the following:

Claim.

$$\Big(\mathcal{H}^i(\Omega^\bullet_X(\log D))\Big)_x \xrightarrow{\sim} \Big(\mathcal{R}^i j_* \mathbb{C}_{X^*}\Big)_x \ \forall x \in D.$$

Let us consider an open neighbourhood U of x, with holomorphic coordinates satisfying the usual condition

$$U \cap D = \{z_1 \cdots z_k = 0\}. \tag{*}$$

First we compute the right-hand side of the equation in the claim. For $j^{-1}(U) = U^* := U - (U \cap D) \cong (\Delta^*)^k \times \Delta^{n-k}$, where Δ denotes the disc $\{t \in \mathbb{C} : |t| < \epsilon\}$, and $\Delta^* = \Delta - 0$. Since the polycilinder $(\Delta^*)^k \times \Delta^{n-k}$ is homotopy equivalent to $(S^1)^k$, the cohomology of U^* is:

$$\begin{array}{ll} H^1(U^*, \mathbb{C}) & \cong \mathbb{C}\frac{dz_1}{z_1} \oplus \cdots \oplus \mathbb{C}\frac{dz_k}{z_k}, \\ H^p(U^*, \mathbb{C}) & \cong \bigwedge^p H^1(U^*, \mathbb{C}). \end{array}$$

On the other hand, $\mathcal{H}^i(\Omega^\bullet_X(\log D)) \cong (a_p)_* \mathbb{C}_{D^{[p]}}$ $\forall p \geq 0$, and

$$\Big((a_p)_*\mathbb{C}_{D^{[p]}}\Big)_x \cong \bigoplus_{i_1<\dots<i_p} \mathbb{C}_{i_1,\dots,i_p}.$$

This allows us to construct explicitly an isomorphism, by mapping the generator of $\mathbb{C}_{i_1,\dots,i_p}$ to $\frac{\mathrm{d}z_{i_1}}{z_{i_1}} \wedge \dots \wedge \frac{\mathrm{d}z_{i_p}}{z_{i_p}}$. □

Proposition 3.14.

$$\mathbb{H}^i(X, \Omega^\bullet_X(\log D)) \cong H^i(X^*, \mathbb{C}_{X^*}).$$

Proof. We know that $\Omega^\bullet_{X^*}$ is j_*-acyclic (by Cartan's theorem B). Then

$$j_*\Omega^\bullet_{X^*} \xrightarrow{\text{q. is.}} j_*\mathcal{C}^\bullet_{\text{Gdm}}(\mathbb{C}_{X^*}),$$

is again a resolution. Recall the definition of direct image sheaf:

$$\mathcal{R}^\bullet j_*\Omega^\bullet_{X^*} := j_*\mathcal{C}^\bullet_{\text{Gdm}}(\Omega^\bullet_{X^*}).$$

$$\mathcal{R}^\bullet j_*\mathbb{C}_{X^*} := j_*\mathcal{C}^\bullet_{\text{Gdm}}(\mathbb{C}_{X^*}).$$

The situation is as follows:

$$\begin{tikzcd}
\mathbb{C}_{X^*}[0] \arrow[rr, "\text{q. is.}"] \arrow[dr, "\text{q. is.}"'] & & \mathcal{C}^\bullet_{\text{Gdm}}(\Omega^\bullet_{X^*}) \\
& \mathcal{C}^\bullet_{\text{Gdm}}(\mathbb{C}_{X^*}). \arrow[ur, "\text{q. is.}"'] &
\end{tikzcd}$$

Then

$$\begin{array}{ccc}
\Omega^\bullet_X(\log D) & \xrightarrow{\text{q. is.}} & j_*\Omega^\bullet_{X^*} \\
& & \downarrow{\scriptstyle \text{q. is.}} \\
\mathcal{R}^\bullet j_*\mathbb{C}_{X^*} & \xrightarrow{\text{q. is.}} & \mathcal{R}^\bullet j_*\Omega^\bullet_{X^*},
\end{array}$$

which implies

$$\mathbb{H}^k(X, \Omega^\bullet_X(\log D)) \cong H^k(X^*, \mathbb{C}_{X^*}).$$

□

3.5 The Hodge-De Rham complex of (X, D)

We consider now some constructions by Deligne.

Definition 3.15. Let (K^*, d_K) be a bounded below differential complex of objects in some abelian category. The *trivial filtration* on $K^\bullet$ is the decreasing filtration $\sigma^{\geq}$,

$$\sigma^{\geq p} := \{0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow K^p \longrightarrow K^{p+1} \longrightarrow \cdots\}.$$

The *canonical filtration* on $K^\bullet$ is the increasing filtration $\tau_{\leq}$,

$$\tau_{\leq p} := \{K^0 \longrightarrow K^1 \longrightarrow \cdots \longrightarrow K^{p-1} \longrightarrow \ker d_K^p \longrightarrow \cdots\}.$$

$$\mathrm{Gr}_p^{\tau_\leq}(K^\bullet) = \{0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow K^{p-1}/\ker d_K^{p-1} \longrightarrow \ker d_K^p \longrightarrow \cdots\}.$$

There is a canonical map

$$\mathrm{Gr}_p^{\tau_\leq}(K^\bullet) \longrightarrow H^p(K^\bullet)[-p],$$

which is a quasi-isomorphism because the diagram

$$\begin{tikzcd} K^{p-1}/\ker d_K^{p-1} \arrow[r, equal, "\sim"] \arrow[dr] & \mathrm{Im}\, d_K^{p-1} \arrow[d, hook] \\ & \ker d_K^p \end{tikzcd}$$

commutes.

As a consequence, we have

$$H^p(\mathrm{Gr}_p^{\tau_\leq}(K^\bullet)) \cong H^p(K^\bullet).$$

If $K^\bullet \longrightarrow L^\bullet$ is a quasi-isomorphism, then the filtered morphism of complexes $(K^\bullet, \tau_\leq) \longrightarrow (L^\bullet, \tau_\leq)$ is a *filtered quasi-isomorphism*, i. e.,

$$\mathrm{Gr}_p^{\tau_\leq}(K^\bullet) \longrightarrow \mathrm{Gr}_p^{\tau_\leq}(L^\bullet) \text{ is a quasi-isomorphism } \forall p.$$

We want to apply this to the De Rham complex of logarithmic forms. The inclusion

$$(\Omega_X^\bullet(\log D), \tau_\leq) \hookrightarrow (\Omega_X^\bullet(\log D), W)$$

is a filtered morphism, because by definition $W_m\Omega_X^\bullet(\log D) = \Omega_X^p(\log D)$ for $m \geq p$:

$$\begin{array}{ccccccccc} \Omega_X^0(\log D) & \rightarrow \cdots \rightarrow & \Omega_X^{m-1}(\log D) & \rightarrow & \Omega_X^m(\log D) & \rightarrow & \Omega_X^{m+1}(\log D) & \rightarrow & \cdots \\ \| & & \| & & \cup & & \cup & & \\ \Omega_X^0(\log D) & \rightarrow \cdots \rightarrow & \Omega_X^{m-1}(\log D) & \rightarrow & \ker \mathrm{d} & \rightarrow & 0 & \rightarrow & \cdots \end{array}$$

In the special case $m = 0$:

$$\begin{array}{ccccc} \Omega_X^0(\log D) & \longrightarrow & \Omega_X^1(\log D) & \longrightarrow & \cdots \\ \cup & & \cup & & \\ \mathbb{C} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

$(\Omega^\bullet_X(\log D), \tau_\leq) \longrightarrow (\Omega^\bullet_X(\log D), W)$ is a filtered quasi-isomorphism, because $\mathrm{Gr}^{\tau_\leq}_m(\Omega^\bullet_X(\log D))$ has cohomology only in degree m, and

$$\mathcal{H}^m(\mathrm{Gr}^{\tau_\leq}_m(\Omega^\bullet_X(\log D))) \cong \mathcal{H}^m(\Omega^\bullet_X(\log D)) \cong \mathcal{H}^m(\mathrm{Gr}^W_m \Omega^\bullet_X(\log D)).$$

Moreover,

$$\Omega^\bullet_X(\log D) \xrightarrow{\text{q. is.}} j_*\Omega^\bullet_{X^*} \;\Rightarrow\; (\Omega^\bullet_X(\log D), \tau_\leq) \xrightarrow{\text{q. is.}} (j_*\Omega^\bullet_{X^*}, \tau_\leq),$$

and

$$(j_*\Omega^\bullet_{X^*}, \tau_\leq) \xrightarrow{\text{q. is.}} (\mathcal{R}^\bullet j_*\Omega^\bullet_{X^*}, \tau_\leq) \xleftarrow{\text{q. is.}} (\mathcal{R}^\bullet j_*\mathbb{C}_{X^*}, \tau_\leq).$$

Definition 3.16. Let X be a compact complex manifold, with Kähler metric, and $D \subset X$ a divisor with normal crossings.
The *Hodge-De Rham complex* $\mathcal{K}^\bullet_{DR}(X \log D)$ associated to the couple (X, D) is the given of three data:

1. $\mathcal{R}^\bullet j_*\mathbb{Z}_{X^*} := j_*\mathcal{C}^\bullet_{\mathrm{Gdm}}(\mathbb{Z}_{X^*})$;
2. $(\mathcal{R}^\bullet j_*\mathbb{Q}_{X^*}, \tau_\leq)$ with the natural map
$$\alpha : \mathcal{R}^\bullet j_*\mathbb{Z}_{X^*} \otimes_{\mathbb{Z}_X} \mathbb{Q}_X \xrightarrow{\text{q. is.}} \mathcal{R}^\bullet j_*\mathbb{Q}_{X^*},$$
which is a quasi-isomorphism of complexes of sheaves;
3. $(\Omega^\bullet_X(\log D), W, F)$, with $F = \sigma^\geq$, and the following chain of filtered quasi-isomorphisms:

$$\begin{tikzcd}
 & & (\mathcal{R}^\bullet j_*\mathbb{Q}_{X^*}, \tau_\leq) \otimes_{\mathbb{Q}_X} \mathbb{C}_X \arrow[d] \\
(\Omega^\bullet_X(\log D), \tau_\leq) \arrow[d, hook] \arrow[dr, hook] & & (\mathcal{R}^\bullet j_*\mathbb{C}_{X^*}, \tau_\leq) \arrow[d] \\
(\Omega^\bullet_X(\log D), W) & (j_*\Omega^\bullet_{X^*}, \tau_\leq) \arrow[r] & (\mathcal{R}^\bullet j_*\Omega^\bullet_{X^*}, \tau_\leq)
\end{tikzcd} \qquad (\beta)$$

Verdier, following Grothendieck, writes (β) as

$$\beta : (\mathcal{R}^\bullet j_*\mathbb{Q}_{X^*}, \tau_\leq) \otimes_{\mathbb{Q}_X} \mathbb{C}_X \xrightarrow{\sim} (\Omega^\bullet_X(\log D), W),$$

considering β as an isomorphism in the *derived category* of bounded below filtered complexes of sheaves of complex vector spaces (it is not in general a map of sheaves). (For the definition of derived category, see [PS].)

Remark 3.17. There is a strong analogy between the definition of the Hodge-De Rham complex and the general definition of a mixed Hodge structure. (1) is analogous to the choice of the lattice, (2) to that of the rational weight filtration and (3) to the Hodge filtration. In fact, as we will see later, $\mathcal{K}^\bullet_{DR}(X \log D)$ is what is called a *mixed Hodge complex of sheaves*.

All maps considered in $\mathcal{K}^\bullet_{DR}(X \log D)$ are filtered morphisms. Then they induce morphisms on the corresponding gradeds, maintaining the same properties. In this way we obtain $\forall m$ the triple

$$(\operatorname{Gr}^{\tau_\leq}_m \mathcal{R}^\bullet j_*\mathbb{Z}_{X^*}, (\operatorname{Gr}^W_m \Omega^\bullet_X(\log D), F), \operatorname{Gr}_m(\beta)),$$

where

$$\operatorname{Gr}_m(\beta) : \operatorname{Gr}^{\tau_\leq}_m \mathcal{R}^\bullet j_*\mathbb{Z}_{X^*} \otimes_{\mathbb{Z}_X} \mathbb{C}_X \longrightarrow \operatorname{Gr}^W_m \Omega^\bullet_X(\log D).$$

(We prefer to consider $\mathbb{Z}_X$ instead of $\mathbb{Q}_X$ as was done in (3), in order to obtain a more general result).

Let us consider the action of the Poincaré residue map on this triple.

We have

$$\mathcal{H}^m(\operatorname{Gr}^{\tau_\leq}_m \mathcal{R}^\bullet j_*\mathbb{C}_{X^*}) = \mathcal{H}^m(\mathcal{R}^\bullet j_*\mathbb{C}_{X^*}) = \mathcal{R}^m j_*\mathbb{C}_{X^*}.$$

We already know

$$\mathcal{R}^m j_*\mathbb{C}_{X^*} = \mathbb{H}^m(X, \Omega^\bullet_X(\log D)) = \mathbb{H}^m(X, \operatorname{Gr}^W_m \Omega^\bullet_X(\log D)) \xrightarrow[\text{q. is.}]{\operatorname{Res}_m} (a_m)_*\mathbb{C}_{D^{[m]}}.$$

At the integer level we have:

$$\begin{array}{ccc}
\mathcal{R}^m j_*\mathbb{C}_{X^*} = \mathbb{H}^m(X, \Omega^\bullet_X(\log D)) = \mathbb{H}^m(X, \operatorname{Gr}^W_m \Omega^\bullet_X(\log D)) & \xrightarrow[\text{q. is.}]{\operatorname{Res}_m} & (a_m)_*\mathbb{C}_{D^{[m]}} \\
\uparrow & & \uparrow \\
\mathcal{R}^m j_*\mathbb{Z}_{X^*} & \xrightarrow[\text{isomorphism}]{\sim} & \left(\frac{1}{2\pi i}\right)^m (a_m)_*\mathbb{Z}_{D^{[m]}}.
\end{array}$$

The latter sheaf in the diagram is the $(-m)$-th Tate twist of $(a_m)_*\mathbb{Z}_{D^{[m]}}$,

$$(a_m)_*\mathbb{Z}_{D^{[m]}}(-m) := \left(\frac{1}{2\pi i}\right)^m (a_m)_*\mathbb{Z}_{D^{[m]}}.$$

We show now the commutativity of the diagram. We need only to establish the property locally in a neighbourhood of a point $x \in D$. Let us consider coordinates $(z_1, \ldots, z_n)$ satisfying the usual condition (*). For $m = 1$ we have:

$$\begin{array}{ll}
\left(\mathcal{R}^1 j_*\mathbb{C}_{X^*}\right)_x & = H^1((\Delta^*)^k \times \Delta^{n-k}, \mathbb{C}) = \mathbb{C}\frac{\mathrm{d}z_1}{z_1} \oplus \cdots \oplus \mathbb{C}\frac{\mathrm{d}z_k}{z_k} \\
\cup & \\
\left(\mathcal{R}^1 j_*\mathbb{Z}_{X^*}\right)_x & = H^1((\Delta^*)^k \times \Delta^{n-k}, \mathbb{Z}).
\end{array}$$

We recall that the first homology group of $(\Delta^*)^k \times \Delta^{n-k}$ is

$$H_1((\Delta^*)^k \times \Delta^{n-k}, \mathbb{Z}) = \mathbb{Z}\gamma_1 \oplus \cdots \oplus \mathbb{Z}\gamma_k,$$

where each γ_j $(j = 1, \ldots, k)$ is a loop around the origin in $\Delta_j = \{|z_j| < \epsilon\}$. By the residue formula,

$$\int_{\gamma_j} \frac{\mathrm{d}z_j}{z_j} = 2\pi i,$$

so that $\left\{\frac{1}{2\pi i}\frac{\mathrm{d}z_j}{z_j}\right\}_{j=1,\dots,k}$ represent integral cohomology classes in $H^1((\Delta^*)^k \times \Delta^{n-k}, \mathbb{Z})$. They constitute in fact the dual basis over $\mathbb{Z}$ of the basis $\gamma_1, \dots, \gamma_k$. This proves the claim for $m = 1$.

For general m,

$$\begin{aligned} \mathcal{R}^m j_* \mathbb{Z}_{X^*} &= \bigwedge\nolimits^m \mathcal{R}^1 j_* \mathbb{Z}_{X^*} \\ (a_m)_* \mathbb{Z}_{D^{[m]}} &= \bigwedge\nolimits^m (a_1)_* \mathbb{Z}_{D^{[1]}}. \end{aligned}$$

The Poincaré residue map induces a map

$$\mathrm{Res}_m : \mathrm{Gr}_m^{\tau_\leq}(\mathcal{R}_*^j \mathbb{Z}_{X^*}) \longrightarrow (a_m)_* \mathbb{Z}_{D^{[m]}}[-m](-m)$$

that is a quasi-isomorphism. We have then an isomorphism

$$\mathrm{Res}_m : (\mathrm{Gr}_m^W \Omega_X^\bullet(\log D), F) \xrightarrow{\sim} ((a_m)_* \Omega_{D^{[m]}}^\bullet[-m], F[-m]).$$

Res gives also a map corresponding to Gr_β, because

$$(a_m)_* \mathbb{Z}_{D^{[m]}}[-m](-m) \otimes_{\mathbb{Z}_X} \mathbb{C}_X \cong (a_m)_* \mathbb{C}_{D^{[m]}}[-m] \xrightarrow[\text{q. is.}]{\mathrm{Res}_m} (a_m)_* \Omega_{D^{[m]}}^\bullet[-m].$$

Summarizing, the Poincaré residue map gives a map of triples:

$$\begin{array}{c} (\mathrm{Gr}_m^{\tau_\leq} \mathcal{R}^\bullet j_* \mathbb{Z}_{X^*}, (\mathrm{Gr}_m^W \Omega_X^\bullet(\log D), F), \mathrm{Gr}_m(\beta)) \\ \Big\downarrow \mathrm{Res}_m \\ ((a_m)_* \mathbb{Z}_{D^{[m]}}[-m](-m), ((a_m)_* \Omega_{D^{[m]}}^\bullet[-m], F[-m]), \mathbb{C}_{D^{[m]}}[0] \xrightarrow{\text{q. is.}} \Omega_{D^{[m]}}^\bullet). \end{array}$$

consisting of maps that are isomorphisms in the derived category.

4 The theory of Deligne

The general theory of Deligne deals with a complex $K^\bullet$ endowed with two filtrations W and F. It considers the spectral sequence associated to the filtered complex $(K^\bullet, W)$ and investigates how the filtration F induces a filtration there. This study reveals abstract properties ensuring that various hypercohomology groups of a mixed Hodge complex of sheaves yield a mixed Hodge structure. A general principle is then as follows: In order to get a mixed Hodge structure on hypercohomology of some geometric object, it suffices to construct a suitable mixed Hodge complex of sheaves on this object.

4.1 Integral Hodge complex of sheaves

We will see now that the structures naturally arisen in the previous section are special cases of more general concepts.

Definition 4.1 ([HIII],[PS]). A *$\mathbb{Z}$-Hodge complex of weight m* is the given of the following:

1. A bounded below complex of $\mathbb{Z}$-modules $K_{\mathbb{Z}}^{\bullet}$ such that $\operatorname{rank} H^k(K_{\mathbb{Z}}^{\bullet}) < \infty\ \forall k$.
2. A bounded below complex $K_{\mathbb{C}}^{\bullet}$ of $\mathbb{C}$-vector spaces equipped with a decreasing filtration F and a comparison morphism

$$\alpha : K_{\mathbb{Z}}^{\bullet} \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} K_{\mathbb{C}}^{\bullet},$$

which is an isomorphism in the derived category.

The data $(K_{Z}^{\bullet}, (K_{C}^{\bullet}, F))$ have moreover to satisfy the following condition, called *Hodge completion axiom*:

(HC) (i) $\forall k \geq 0$, $(H^k(K_{\mathbb{Z}}^{\bullet}), (H^k(K_{\mathbb{C}}^{\bullet}), F))$ is an integral Hodge structure of weight $k+m$;

(ii) the spectral sequence of $(K_{\mathbb{C}}^{\bullet}, F)$ degenerates at E_1; (equivalently, the differential of the complex $K_{\mathbb{C}}^{\bullet}$ is strictly compatible with the filtration F).

Definition 4.2 ([HIII],[PS]). A *$\mathbb{Z}$-Hodge complex of sheaves on X of weight m*, where X is a topological space, is the given of the following:

1. A bounded below complex $\mathcal{K}_{\mathbb{Z}}^{\bullet}$ of sheaves of $\mathbb{Z}$-modules on X, such that $\operatorname{rank} \mathbb{H}^k(X, \mathcal{K}_{\mathbb{Z}}^{\bullet}) < \infty\ \forall k$.
2. A bounded below complex $\mathcal{K}_{\mathbb{Z}}^{\bullet}$ of sheaves of $\mathbb{C}$-vector spaces, equipped with a decreasing filtration F and a comparison morphism

$$\alpha : \mathcal{K}_{\mathbb{Z}}^{\bullet} \otimes_{\mathbb{Z}_X} \mathbb{C}_X \xrightarrow{\sim} \mathcal{K}_{\mathbb{C}}^{\bullet},$$

which is an isomorphism in the derived category.

The data $(\mathcal{K}_{\mathbb{Z}}^{\bullet}, (\mathcal{K}_{\mathbb{C}}^{\bullet}, F))$ have to satisfy the following condition (*Hodge completion axiom for complexes of sheaves*):

(HCS) (i) $(\mathbb{H}^k(X, \mathcal{K}_{\mathbb{Z}}^{\bullet}), (\mathbb{H}^k(X, \mathcal{K}_{\mathbb{C}}^{\bullet}), F))$ is an integral Hodge structure of weight $k+m$;

(ii) the spectral sequence of $(\mathcal{R}\Gamma(X, \mathcal{K}_{\mathbb{C}}^{\bullet}), F)$ degenerates at E_1.

Remark 4.3. There is a natural relation between the two definitions:

$$(\mathcal{K}_{\mathbb{Z}}^{\bullet}, (\mathcal{K}_{\mathbb{C}}^{\bullet}, F), \alpha) \underset{\mathcal{R}\Gamma(X,-)}{\overset{\text{functor}}{\rightsquigarrow\rightsquigarrow\rightsquigarrow}} (\mathcal{R}\Gamma(X, \mathcal{K}_{\mathbb{Z}}^{\bullet}), (\mathcal{R}\Gamma(X, \mathcal{K}_{\mathbb{C}}^{\bullet}), F), \mathcal{R}\Gamma(X, \alpha))$$

because, by definition,

$$\mathbb{H}^k(X, \mathcal{K}_{\mathbb{Z}}^{\bullet}) = H^k \mathcal{R}\Gamma(X, \mathcal{K}_{\mathbb{Z}}^{\bullet}),$$

$$\mathcal{K}_{\mathbb{C}}^{\bullet} \xrightarrow{\text{q. is.}} \mathcal{C}_{\text{Gdm}}^{\bullet}(\mathcal{K}_{\mathbb{C}}^{\bullet})$$

and

$$\mathcal{R}\Gamma(X, \mathcal{K}_{\mathbb{C}}^{\bullet}) = \Gamma(X, \mathcal{C}_{\text{Gdm}}^{\bullet}(\mathcal{K}_{\mathbb{C}}^{\bullet})).$$

We recall here that $\Gamma(X, \mathcal{C}_{\text{Gdm}}^{\bullet}(-))$ is an exact functor, for the same reasons why $\mathcal{C}_{\text{Gdm}}^{\bullet}(-)$ is (see [G]).

Then $\mathcal{R}\Gamma(X, \mathcal{K}_{\mathbb{C}}^{\bullet})$ is filtered by $\mathcal{R}\Gamma(X, F^p\mathcal{K}_{\mathbb{C}}^{\bullet})$.

Remark 4.4. As in the case of mixed Hodge structures, we can define $\mathbb{Q}$- (respectively, $\mathbb{R}$- or $\mathbb{C}$-) *Hodge complexes* and *complexes of sheaves* by substituting $\mathbb{Z}$ in the definition with $\mathbb{Q}$ (respectively, $\mathbb{R}$ or $\mathbb{C}$).

Example. Let X be a complex compact Kähler manifold. The Hodge-De Rham complex of sheaves on X is defined by

$$\mathcal{K}^\bullet_{DR}(X) = (\mathbb{Z}_X[0], (\Omega^\bullet_X, \sigma^{\geq}), \mathbb{C}_X[0] \xrightarrow{\text{q. is.}} \Omega^\bullet_X).$$

(The inclusion $\mathbb{C}_X[0] \hookrightarrow \Omega^\bullet_X$ is a quasi-isomorphism by the holomorphic Poincaré lemma.)

$\mathcal{K}^\bullet_{DR}(X)$ is a $\mathbb{Z}$-Hodge complex of sheaves of weight 0. Let us verify the axiom (HCS).

We have to prove that the filtration

$$F^p\mathbb{H}^k(X, \Omega^\bullet_X) := \operatorname{Im}(\mathbb{H}^k(X, \sigma^{\geq p}\Omega^\bullet_X) \longrightarrow \mathbb{H}^k(X, \Omega^\bullet_X)))$$

is a Hodge filtration on $\mathbb{H}^k(X, \Omega^\bullet_X)$. We will do it later, by showing that it coincides with the usual Hodge filtration F^p_H on $H^\bullet_{DR}(X, \mathbb{C})$.

Let us consider the spectral sequence of $(\mathcal{R}\Gamma(X, \Omega^\bullet_X), \sigma^{\geq})$.

$$E_1^{p,q} = \mathbb{H}^{p+q}(X, \operatorname{Gr}_p^{\sigma^{\geq}} \Omega^\bullet_X) = \mathbb{H}^{p+q}(X, \Omega^X_p[p]) = H^q(X, \Omega^X_p) \cong H^{p,q}(X),$$

the first element in the spectral sequence is isomorphic to the space of harmonic (p, q)-forms.

$$E_\infty^{p,q} = \operatorname{Gr}_p^F \mathbb{H}^{p+q}(X, \Omega^\bullet_X) = \operatorname{Gr}_p^F H^{p+q}_{DR}(X, \mathbb{C}).$$

As a consequence,

$$\sum_{p+q=k} \dim E_1^{p,q} = \sum_{p+q=k} \dim H^{p,q}(X) = \dim_{\mathbb{C}} H^k_{DR}(X, \mathbb{C}) = \sum_{p+q=k} \dim E_\infty^{p,q}.$$

This implies that E_1 and E_∞ have the same dimension; hence, as for the recursive property of the spectral sequence E_{r+1} is a subquotient of E_r, the spectral sequence itself degenerates at E_1. This is equivalent to say that the map

$$\mathbb{H}^k(X, \sigma^{\geq p}\Omega^\bullet_X) \longrightarrow \mathbb{H}^k(X, \Omega^\bullet_X)$$

is injective $\forall k$. Let us see why this is true in the simple case of a complex of sheaves $\mathcal{K}^\bullet$ has only two non-trivial terms.

$$\begin{array}{rlllllll} \mathcal{K}^\bullet & := \{0 & \to & \mathcal{K}^0 & \to & \mathcal{K}^1 & \to & 0\}, \\ \sigma^{\geq 1}\mathcal{K}^\bullet & := \{0 & \to & 0 & \to & \mathcal{K}^1 & \to & 0\}. \end{array}$$

Then we have the exact sequence

$$0 \longrightarrow \sigma^{\geq 1}\mathcal{K}^\bullet \longrightarrow \mathcal{K}^\bullet \longrightarrow \mathcal{K}^\bullet/\sigma^{\geq 1}\mathcal{K}^\bullet \longrightarrow 0,$$

where $\mathcal{K}^\bullet/\sigma^{\geq 1}\mathcal{K}^\bullet := \{0 \to \mathcal{K}^0 \to 0 \to 0\}$. The associated long exact sequence is:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{H}^{q-1}(X, \mathcal{K}^\bullet/\sigma^{\geq 1}\mathcal{K}^\bullet) \arrow[r] \arrow[d, equal] & \mathbb{H}^q(X, \sigma^{\geq 1}\mathcal{K}^\bullet) \arrow[r, "(*)"] \arrow[d, equal] & \mathbb{H}^q(X, \mathcal{K}^\bullet) \arrow[r] & \cdots \\
& H^{q-1}(\mathcal{K}^0) \arrow[r, "d_1"] & H^{q-1}(\mathcal{K}^1) & &
\end{tikzcd}$$

which implies that $d_1 = 0$ if and only if that the map $(*)$ is injective.

If the complex $\mathcal{K}^\bullet$ is longer, then we can factorize the map on the hypercohomology by a chain of injective maps.

$$\begin{tikzcd}
\mathbb{H}^k(X, \sigma^{\geq p}\Omega_X^\bullet) \arrow[rrr] \arrow[d, "\text{injective}"] & & & \mathbb{H}^k(X, \Omega_X^\bullet) \\
\mathbb{H}^k(X, \sigma^{\geq p+1}\Omega_X^\bullet) \arrow[r, "\text{injective}"'] & \mathbb{H}^k(X, \sigma^{\geq p+2}\Omega_X^\bullet) \arrow[r, "\text{injective}"'] & \cdots\cdots \arrow[ur, "\text{injective}"'] &
\end{tikzcd}$$

$E_1 = E_\infty$ implies

$$h^{p,k-p}(X) := \dim H^{p,k-p} = \dim \mathbb{H}^k(X, \sigma^{\geq p}\Omega_X^\bullet) - \dim \mathbb{H}^k(X, \sigma^{\geq p+1}\Omega_X^\bullet)$$

(the dimension of the cohomology of the graded equals that of the graded of the cohomology), and, eventually,

$$\dim \mathbb{H}^k(X, \sigma^{\geq p}\Omega_X^\bullet) = \sum_{r \geq p} h^{r,k-r}(X) = \dim F_H^p H_{DR}^k(X, \mathbb{C}),$$

where F_H denotes the usual Hodge filtration on the De Rham cohomology.

We can prove now by induction the following

Claim.
$$F^p\mathbb{H}^k(X, \Omega_X^\bullet) = F_H^p H_{DR}^k(X, \mathbb{C}) =: \bigoplus_{r \geq p} H^{r,k-r}(X).$$

Proof of claim. The claim holds for $p = 0$.

For $p = 1$, we have

$$\begin{aligned}
F_H^1 H_{DR}^k(X, \mathbb{C}) &= \ker(H_{DR}^k(X, \mathbb{C}) \longrightarrow H^{0,k}(X)) = \\
&= \ker(\mathbb{H}^k(X, \Omega_X^\bullet) \longrightarrow H^k(X, \mathcal{O}_X)) = \\
&= \ker(\mathbb{H}^k(X, \Omega_X^\bullet) \longrightarrow \underset{\substack{\| \\ \mathbb{H}^k(X, \mathcal{O}_X[0])}}{\mathbb{H}^k(X, \Omega_X^\bullet/\sigma^{\geq 1}\Omega_X^\bullet)}),
\end{aligned}$$

where we have used both the De Rham and the Dolbeault isomorphisms (on the two sides of the projection).

The long exact sequence splits into the exact sequence

$$0 \to \sigma^{\geq 1}\Omega_X^\bullet \to \Omega_X^\bullet \to \Omega_X^\bullet/\sigma^{\geq 1}\Omega_X^\bullet \to 0,$$

which gives us

$$F_H^1 H_{DR}^k(X,\mathbb{C}) = \mathbb{H}^k(X, \sigma^{\geq 1}\Omega_X^\bullet).$$

This is no longer a mere identity of dimensions, but a genuine identity of subspaces.

The general case $p \geq 2$ is proved analogously. $\square$

Let us consider a triple $(K^\bullet, F, \mathrm{d})$, where $K^\bullet$ is a bounded below complex with differential $\mathrm{d} : K^\bullet \longrightarrow K^{\bullet+1}$, and F is a decreasing filtration.

Proposition 4.5. *The following are equivalent:*

1. *the spectral sequence degenerates at E_1;*
2. *the map $H^k(F^pK^\bullet) \longrightarrow H^k(K^\bullet)$ is injective $\forall k$;*
3. d *is strictly compatible with the filtration F.*

Proof. We proved the equivalence of (1) and (2). We give here a direct proof of the equivalence of (3) and (2).

Let us consider the map $H^k(F^pK^\bullet) \longrightarrow H^k(K^\bullet)$ for some index k. Then it is injective if and only if for all $\alpha \in F^pK^k$ such that $\alpha = \mathrm{d}\beta$, $\beta \in K^{k-1}$, there exists $\beta' \in F^pK^{k-1}$ such that $\alpha = \mathrm{d}\beta'$. This is precisely equivalent to $F^pK^k \cap \mathrm{d}K^{k-1} = \mathrm{d}(F^pK^{k-1})$. $\square$

Definition 4.6. Let $\mathcal{K}^\bullet = (\mathcal{K}_{\mathbb{Z}}^\bullet, (\mathcal{K}_{\mathbb{C}}^\bullet, F), \alpha)$ be a $\mathbb{Z}$-Hodge complex of sheaves of weight m on X.
Then for all $n \in \mathbb{Z}$ we define the *n-th Tate twist of* $\mathcal{K}^\bullet$,

$$\mathcal{K}^\bullet(n) = (\mathcal{K}_{\mathbb{Z}}^\bullet(n), (\mathcal{K}_{\mathbb{C}}^\bullet, F[n]), \alpha)$$

by taking $\mathcal{K}_{\mathbb{Z}}^\bullet(n) := (2\pi i)^n \mathcal{K}_{\mathbb{Z}}^\bullet$ and the usual shifted filtration $(F[n])^p = F^{p+n}$.

The effect of the Tate twist on the hypercohomology of the complex is that we have

$$\mathbb{H}^k(X, \mathcal{K}_{\mathbb{C}}^\bullet(n)) = \mathbb{H}^k(X, \mathcal{K}_{\mathbb{C}}^\bullet)(n) = \mathbb{H}^k(X, \mathcal{K}_{\mathbb{C}}^\bullet) \otimes_{\mathbb{C}} \mathbb{C}(n),$$

where

$$\begin{cases} \mathbb{Z}(n) & := (2\pi i)^n \mathbb{Z} \subset \mathbb{C} \\ \mathbb{C}(n) & := \bigoplus_{p,q \in \mathbb{Z}} H^{p,q}, \ H^{p,q} := \begin{cases} \mathbb{C} & \text{if } p = q = -n, \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

is the *Tate-Hodge structure on* $\mathbb{C}$ *of weight* $-2n$.

This proves that $\mathcal{K}^\bullet(n)$ is a $\mathbb{Z}$-Hodge complex of sheaves of weight $m - 2n$.

Definition 4.7. Let $\mathcal{K}^\bullet = (\mathcal{K}_{\mathbb{Z}}^\bullet, (\mathcal{K}_{\mathbb{C}}^\bullet, F), \alpha)$ be a $\mathbb{Z}$-Hodge complex of sheaves of weight m on X.
Then for all $r \in \mathbb{Z}$ we define the *shift of* $\mathcal{K}^\bullet$ *by* r as

$$\mathcal{K}^\bullet[r] := (\mathcal{K}_{\mathbb{Z}}^\bullet[r], (\mathcal{K}_{\mathbb{C}}^\bullet[r], F), \alpha).$$

In this case $\mathbb{H}^k(X, \mathcal{K}_{\mathbb{Z}}^\bullet[r]) = \mathbb{H}^{k+r}(X, \mathcal{K}_Z^\bullet)$: $\mathcal{K}^\bullet[r]$ is a $\mathbb{Z}$-Hodge complex of sheaves of weight $m + r$.

4.2 Integral mixed Hodge complex of sheaves

Definition 4.8 ([HIII],[PS]). A *mixed $\mathbb{Z}$-Hodge complex* is the given of the following:

1. A bounded below complex of $\mathbb{Z}$-modules $K^\bullet_\mathbb{Z}$, with $\operatorname{rank} H^k(K^\bullet_\mathbb{Z}) < \infty$.
2. A bounded below complex $K^\bullet_\mathbb{Q}$ of $\mathbb{Q}$-vector spaces equipped with a increasing filtration W and a comparison morphism
$$\alpha : K^\bullet_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q} \xrightarrow{\sim} K^\bullet_\mathbb{Q},$$
which is an isomorphism in the derived category.
3. A bounded below complex $K^\bullet_\mathbb{C}$ of $\mathbb{C}$-vector spaces equipped with an increasing filtration W, a decreasing filtration F and a comparison morphism
$$\beta : (K^\bullet_\mathbb{Q}, W) \otimes_\mathbb{Q} \mathbb{C} \xrightarrow{\sim} (K^\bullet_\mathbb{C}, W),$$
which is a map in the derived category such that $\forall m$ the induced map
$$\mathrm{Gr}^W_m(\beta) : \mathrm{Gr}^W_m(K^\bullet_\mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C} \xrightarrow{\sim} \mathrm{Gr}^W_m(K^\bullet_\mathbb{C})$$
is an isomorphism in the derived category.

The data $(K^\bullet_\mathbb{Z}, (K^\bullet_\mathbb{Q}, W), \alpha, (K^\bullet_\mathbb{C}, W, F), \beta)$ have moreover to satisfy the following condition:

(*) $\forall m$, $(\mathrm{Gr}^W_m(K^\bullet_\mathbb{Q}), (\mathrm{Gr}^W_m(K^\bullet_\mathbb{C}), F), \mathrm{Gr}^W_m(\beta))$ is a $\mathbb{Q}$-Hodge complex of weight m.

Definition 4.9 ([HIII],[PS]). A *mixed $\mathbb{Z}$-Hodge complex of sheaves* on a topological space X is the given of the following:

1. A bounded below complex $\mathcal{K}^\bullet_\mathbb{Z}$ of sheaves of $\mathbb{Z}$-modules on X, such that $\operatorname{rank} \mathbb{H}^k(X, \mathcal{K}^\bullet_\mathbb{Z}) < \infty$.
2. A bounded below complex $\mathcal{K}^\bullet_\mathbb{Q}$ of sheaves of $\mathbb{Q}$-vector spaces on X equipped with a increasing filtration W and a comparison morphism
$$\alpha : \mathcal{K}^\bullet_\mathbb{Z} \otimes_{\mathbb{Z}_X} \mathbb{Q}_X \xrightarrow{\sim} \mathcal{K}^\bullet_\mathbb{Q},$$
which is an isomorphism in the derived category.
3. A bounded below complex $\mathcal{K}^\bullet_\mathbb{C}$ of sheaves of $\mathbb{C}$-vector spaces on X equipped with an increasing filtration W, a decreasing filtration F and a comparison morphism
$$\beta : (\mathcal{K}^\bullet_\mathbb{Q}, W) \otimes_{\mathbb{Q}_X} \mathbb{C}_X \xrightarrow{\sim} (\mathcal{K}^\bullet_\mathbb{C}, W),$$
which is a map in the derived category such that $\forall m$ the induced map
$$\mathrm{Gr}^W_m(\beta) : \mathrm{Gr}^W_m(\mathcal{K}^\bullet_\mathbb{Q}) \otimes_{\mathbb{Q}_X} \mathbb{C}_X \xrightarrow{\sim} \mathrm{Gr}^W_m(\mathcal{K}^\bullet_\mathbb{C})$$
is an isomorphism in the derived category.

The data $(\mathcal{K}^\bullet_\mathbb{Z}, (\mathcal{K}^\bullet_\mathbb{Q}, W), \alpha, (\mathcal{K}^\bullet_\mathbb{C}, W, F), \beta)$ have moreover to satisfy the following condition:

(**) $\forall m$, $(\mathrm{Gr}^W_m(\mathcal{K}^\bullet_\mathbb{Q}), (\mathrm{Gr}^W_m(\mathcal{K}^\bullet_\mathbb{C}), F), \mathrm{Gr}^W_m(\beta))$ is a $\mathbb{Q}$-Hodge complex of sheaves of weight m.

Example. Let X be a compact Kähler manifold, and $D \subset X$ a normal crossing divisor. Then the Hodge-De Rham complex associated to (X, D) is

$$\mathcal{K}^\bullet_{DR}(X \log D) = (\mathcal{R}^\bullet j_* \mathbb{Z}_{X^*}, (\mathcal{R}^\bullet j_* \mathbb{Q}_{X^*}, \tau_\leq), \alpha, (\Omega^\bullet_X(\log D), W, F), \beta),$$

where $X^* = X - D$, $j : X^* \hookrightarrow X$.

As was shown in section 3.1, the Poincaré residue map gives a map of triples:

$$\begin{array}{c}(\mathrm{Gr}^{\tau_\leq}_m \mathcal{R}^\bullet j_* \mathbb{Q}_{X^*}, (\mathrm{Gr}^W_m \Omega^\bullet_X(\log D), F), \mathrm{Gr}_m(\beta)) \\ \Big\downarrow{\scriptstyle \mathrm{Res}_m} \\ ((a_m)_* \mathbb{Q}_{D^{[m]}}[-m](-m), ((a_m)_* \Omega^\bullet_{D^{[m]}}[-m], F[-m]), \mathbb{C}_{D^{[m]}}[0] \xrightarrow{\text{q. is.}} \Omega^\bullet_{D^{[m]}}).\end{array}$$

constituted by maps that are isomorphisms in the derived category.

By Deligne's results, the collection of data

$$(\mathbb{Q}_{D^{[m]}}[0], (\Omega^\bullet_{D^{[m]}}[0], F), \mathbb{C}_{D^{[m]}}[0] \xrightarrow{\text{q. is.}} \Omega^\bullet_{D^{[m]}})$$

is a $\mathbb{Q}$-Hodge complex of sheaves of weight 0. (Recall that $D^{[m]}$ is compact Kähler. This implies that $\mathrm{Gr}^W_m \mathcal{K}^\bullet_{DR}(X \log D)$ is a $\mathbb{Q}$-Hodge complex of sheaves of weight $-m + 2m = m$.

4.3 The fundamental theorem of Deligne

Theorem 4.10 (Deligne). Fundamental theorem.
Let $(\mathcal{K}^\bullet_\mathbb{Z}, (\mathcal{K}^\bullet_\mathbb{Q}, W), \alpha, (\mathcal{K}^\bullet_\mathbb{C}, W, F), \beta)$ be a mixed $\mathbb{Z}$-Hodge complex of sheaves on a topological space X. Pose

$$\begin{array}{ll} K^\bullet_\mathbb{Z} & := \mathcal{R}\Gamma(\mathcal{K}^\bullet_\mathbb{Z}), \\ K^\bullet_\mathbb{Q} & := \mathcal{R}\Gamma(\mathcal{K}^\bullet_\mathbb{Q}), \\ K^\bullet_\mathbb{C} & := \mathcal{R}\Gamma(\mathcal{K}^\bullet_\mathbb{C}), \end{array}$$

and denote again with F, W the induced filtrations and with α, β the induced comparison morphisms. Then we have:

1. *The triple $(K^\bullet_\mathbb{Z}, (K^\bullet_\mathbb{Q}, W), \alpha, (K^\bullet_\mathbb{C}, W, F), \beta)$ is a mixed $\mathbb{Z}$-Hodge complex.*
2. *The filtrations* Dec W*, defined as $W[k]$ on the cohomology in degree k, and F, induce a mixed Hodge structure on $\mathbb{H}^k(X, \mathcal{K}^\bullet_\mathbb{Z})$. In fact we can say*
   (a) *the filtration $W[k]$ on $\mathbb{H}^k(X, \mathcal{K}^\bullet_\mathbb{Q})$ and the filtration induced by F on $\mathbb{H}^k(X, \mathcal{K}^\bullet_\mathbb{C})$ define a mixed Hodge structure on $\mathbb{H}^k(X, \mathcal{K}^\bullet_\mathbb{Z})$;*

(b) *the first differential of the spectral sequence associated to* $(\mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{Q}}), W)$ *is strictly compatible with the filtration induced by* F *on* $E_1(\mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{Q}}), W)$*;*

(c) *the spectral sequence* $E_r(\mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{Q}}), W)$ *degenerates at* E_2*;*

(d) *the spectral sequence* ${}_F E_r$ *associated to* $(\mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{C}}), F)$*,*

$$ {}_F E_1^{p,q} = \mathbb{H}^{p+q}(X, \operatorname{Gr}^p_F \mathcal{K}^\bullet_{\mathbb{C}}) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{K}^\bullet_{\mathbb{C}}), $$

degenerates at ${}_F E_1$*. Equivalently,*

$$ \mathbb{H}^k(X, F^p \mathcal{K}^\bullet_{\mathbb{C}}) \longrightarrow \mathbb{H}^k(X, \mathcal{K}^\bullet_{\mathbb{C}}) $$

is always injective, and

$$ \operatorname{Gr}^p_F \mathbb{H}^k(X, \mathcal{K}^\bullet_{\mathbb{C}}) \cong \mathbb{H}^k(X, \operatorname{Gr}^p_F \mathcal{K}^\bullet_{\mathbb{C}}). $$

(e) *the spectral sequence* $E_r(\operatorname{Gr}^p_F \mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{C}}), W)$ *degenerates at* E_2*.*

3. *Any morphism of mixed Hodge complexes of sheaves on* X *induces a morphism of mixed Hodge structures on the hypercohomology groups.*

Application. Let U be a smooth quasi-projective algebraic variety over $\mathbb{C}$. There exists an embedding $U \hookrightarrow X$ in a smooth projective variety X such that $X - U = D$ is a divisor with normal crossings on X. The Hodge-De Rham complex of (X, D) is a mixed Hodge complex of sheaves, and its hypercohomology is isomorphic to the cohomology $H^*(U)$ of U (see section 4). Thus $H^*(U)$ carries an integral mixed Hodge structure.

4.3.1 Useful lemmas

We begin with some useful lemmas.

Remark 4.11. Let (A, F) be a filtered object, and denote by X a subobject of A.

1. The *induced filtration on* X *as a subobject of* A is defined by:

$$ F^n(X) = j^{-1}(F^n(A)) = X \cap F^n(A), $$

where we are considering the immersion $j : X \hookrightarrow A$.

2. Dually, the *induced filtration on* A/X *as a subobject of* A is defined by

$$ F^n(A/X) = p(F^n(A)) = F^n(A)/(X \cap F^n(A)), $$

where p denotes the projection $p : A \twoheadrightarrow A/X$.

Lemma 4.12 (Deligne, [HII]). *Let (A, F) be a filtered object. Suppose $X \subset Y \subset A$. Denote by $'F$ the filtration $'F$ on Y/X as* subobject of a quotient *of A,*

$$(A, F) \xRightarrow{A \twoheadrightarrow A/X} (A/X, F) \xRightarrow{A/X \hookrightarrow Y/X} (Y/X, 'F) \ ,$$

and by $''F$ the filtration on Y/X as quotient of a subobject *of A,*

$$(A, F) \xRightarrow{A \hookrightarrow Y} (Y, F) \xRightarrow{A \twoheadrightarrow A/X} (Y/X, '' F) \ .$$

Then $'F ='' F$.

Lemma 4.13. *Let $(K^\bullet, F)$ be a differential complex equipped with a biregular filtration. Then the following are equivalent:*

1. *the differential d_K of $K^\bullet$ is strictly compatible with F, i.e., $\forall p, n$ we have*
$$d_K(F^p(K^n)) = \mathrm{Im}(d_K) \cap F^p(K^{n+1}).$$
2. *$\forall p, n$, the sequence*
$$0 \to H^n(F^p(K)) \to H^n(K) \to H^n(K/F^pK) \to 0$$
is exact
3. *the spectral sequence $E_r(K^\bullet, F)$ degenerates at E_1.*

For a proof, see [PS, Appendix A].

Lemma 4.14. *Let $L^\bullet$ be a differential complex with only 3 non-trivial terms, filtered by W. Then the complex $\mathrm{Gr}^W_\bullet(L^\bullet)$ is exact if and only if $L^\bullet$ is exact and the differential of $L^\bullet$ is strictly compatible with W.*

4.3.2 Comparison of the three filtrations

In the proof of the Fundamental Theorem we deal with a complex $(K^\bullet, W, F)$. We consider the spectral sequence associated to $(K^\bullet, W)$ and we investigate how F induces a filtration there. The situation is that there are 3 filtrations on $E_r(K^\bullet, W)$, induced by F in a canonical way.

Definition 4.15. Let us consider the exact sequence

$$0 \to F^pK^\bullet \to K^\bullet \to K^\bullet/F^pK^\bullet \to 0.$$

The filtered morphism $(F^pK^\bullet, W) \hookrightarrow (K^\bullet, W)$ induces a map $E_r(F^pK^\bullet, W) \to E_r(K^\bullet, W)$ which is injective for $r = 0$.

Dually, the filtered morphism $(K^\bullet/F^pK^\bullet, W) \twoheadrightarrow (K^\bullet, W)$ induces a map $E_r(K^\bullet/F^pK^\bullet, W) \to E_r(K^\bullet, W)$ which is surjective for $r = 0$. The *first direct filtration* ${}^IF_d$ is defined by

$${}^IF_dE_r(K^\bullet, W) = \mathrm{Im}(E_r(F^pK^\bullet, W) \to E_r(K^\bullet, W)).$$

The *second direct filtration* ${}^{II}F_d$ is defined by

$${}^{II}F_dE_r(K^\bullet, W) = \ker(E_r(K^\bullet, W) \to E_r(K^\bullet/F^pK^\bullet, W)).$$

Remark 4.16. Recall that $E_r^{p,q} = Z_r^{p,q}/(B_r^{p,q} \cap Z_r^{p,q})$. Then ${}^I F_d$ is the same as the filtration obtained by considering $E_r^{p,q}$ as a subquotient of (K^{p+q}, F). As the diagram

$$\begin{tikzcd} Z_r^{p,q} \arrow[r, hook] \arrow[d] & K^{p+q} \arrow[d] \\ Z_r^{p,q}/(B_r^{p,q} \cap Z_r^{p,q}) \arrow[r, hook] & K^{p+q}/B_r^{p,q} \end{tikzcd}$$

commutes, we have

$$E_r^{p,q} \cong \ker(K^{p+q}/B_r^{p,q} \to K^{p+q}/(Z_r^{p,q} + B_r^{p,q})).$$

We see then that ${}^{II} F_d$ is the filtration obtained by considering $E_r^{p,q}$ as a subobject of a quotient of K^{p+q}.

The situation is a variation of that of Lemma 4.12. We have:

$$\begin{tikzcd} & A & \\ Z \arrow[ur, hook] & & B \arrow[ul, hook'] \\ & Z \cap B. \arrow[ul, hook'] \arrow[ur, hook] & \end{tikzcd}$$

Our problem is that $Z/(Z \cap B) \cong (Z + B)/B$ is *not* a filtered isomorphism. So the two subquotients of A are isomorphic but the induced filtration is in general not the same. On the other hand, the map $Z/(Z \cap B) \to (Z + B)/B$ preserve the filtration. Then we know at least that one filtration is contained in the other. In particular, we have ${}^I F_d \subset {}^{II} F_d$.

Definition 4.17 (Deligne). The *inductive filtration* on $E_r(K^\bullet, W)$ is the filtration F_i defined as follows.

For $r = 0, 1$, the first and the second direct filtrations coincide by Lemma 4.12. We define

$$F_i(E_0(K^\bullet, W)) := {}^I F_d(E_0(K^\bullet, W)) = {}^{II} F_d(E_0(K^\bullet, W)).$$

For $r \geq 1$ we use induction. Assume $(E_r(K^\bullet, W), F_i)$ is known, and recall that $E_{r+1}(K^\bullet, W) = H(E_r(K^\bullet, W), d_r)$, so that $E_{r+1}(K^\bullet, W)$ is a subquotient of $E_r(K^\bullet, W)$. Then $(E_{r+1}(K^\bullet, W), F_i)$ is the filtration induced by $(E_r(K^\bullet, W), F_i)$ on the quotient.

Lemma 4.18 (Comparison of the three filtrations).

1. *On E_0, E_1, ${}^I F_d = {}^{II} F_d = F_i$.*
2. *$\forall r < \infty$, ${}^I F_d \subset F_i \subset {}^{II} F_d$.*
3. *Assume W is biregular. Then the filtration F on $E_\infty^{p,q}$ induced by the isomorphism $E_\infty^{p,q} \cong \mathrm{Gr}_{-p}^W H^{p+q}(K^\bullet)$ is related to ${}^I F_d$ and ${}^{II} F_d$ on $E_\infty^{p,q}$ (defined by taking r sufficiently great) by*

$$ {}^I F_d(E_\infty) \subset F(E_\infty) \subset {}^{II} F_d(E_\infty).$$

Proof. (1) is a consequence of Lemma 4.12.

(2) Fix indices $r < \infty, p, q$, and take $s \in \mathbb{Z}$. Then there exists a natural inclusion $\alpha_s : (F^s K^\bullet, W) \hookrightarrow (K^\bullet, W)$ that induces a morphism of spectral sequences

$$\begin{array}{rccc} E_r^{p,q}(\alpha_s): & E_r^{p,q}(F^s K^\bullet, W) & \longrightarrow & E_r^{p,q}(K^\bullet, W) \\ & {\scriptstyle d_r}\big\downarrow & & {\scriptstyle d_r}\big\downarrow \\ & E_r^{p+r,q-r+1}(F^s K^\bullet, W) & \longrightarrow & E_r^{p+r,q-r+1}(K^\bullet, W). \end{array}$$

The image of $E_r^{p,q}(\alpha_s)$ coincides with ${}^I F_d E_r^{p,q}(K^\bullet, W)$ by definition. For the compatibility of $E_r^{p,q}(\alpha_s)$ with the differentials the diagram

$$\begin{tikzcd} {}^I F_d E_r^{p,q}(K^\bullet, W) \arrow[r, hook] \arrow[d, "d_r"] & E_r^{p,q}(K^\bullet, W) \arrow[d, "d_r"] \\ {}^I F_d E_r^{p+r,q-r+1}(K^\bullet, W) \arrow[r, hook] & E_r^{p+r,q-r+1}(K^\bullet, W) \end{tikzcd}$$

commutes, which means that d_r is compatible with ${}^I F_d$.

The inclusion of complexes

$$({}^I F_d^s E_r(K^\bullet, W), d_r) \hookrightarrow (E_r(K^\bullet, W), d_r))$$

induces on cohomology the map

$$H({}^I F_d^s E_r(K^\bullet, W), d_r) \longrightarrow H(E_r(K^\bullet, W), d_r) = E_{r+1}(K^\bullet, W).$$

This suggests to define a filtration $(E_{r+1}(K^\bullet, W), {}^I G)$ such that

$${}^I G^s E_{r+1}(K^\bullet, W) := \operatorname{Im}\left(H({}^I F_d^s E_r(K^\bullet, W), d_r) \longrightarrow H(E_r(K^\bullet, W), d_r)\right).$$

Claim.

$${}^I F_d E_{r+1}(K^\bullet, W) \subset {}^I G E_{r+1}(K^\bullet, W).$$

Proof of the claim. All maps in the diagram

$$\begin{tikzcd} \ker\left(d_r : E_r^{p,q}(F^s K^\bullet, W) \longrightarrow E_r^{p+r,q-r+1}(F^s K^\bullet, W)\right) \arrow[r, two heads] \arrow[d, "E_r^{p,q}(\alpha_s)"] & E_{r+1}^{p,q}(F^s K^\bullet, W) \arrow[d, "{H(E_r^{p,q}(\alpha_s)) = E_{r+1}^{p,q}(\alpha_s)}"'] \\ \ker\left(d_r : E_r^{p,q}(K^\bullet, W) \longrightarrow E_r^{p+r,q-r+1}(K^\bullet, W)\right) \arrow[r, two heads] & E_{r+1}^{p,q}(K^\bullet, W) \end{tikzcd}$$

are well defined, because of the compatibility of ${}^I F_d$ with differentials. The equality $H(E_r^{p,q}(\alpha_s)) = E_{r+1}^{p,q}(\alpha_s)$ is a consequence of the fact that α is compatible with the spectral sequence, and hence also with its structure as an iteration of cohomology. $\ker\left(d_r : E_r^{p,q}(F^s K^\bullet, W) \longrightarrow E_r^{p+r,q-r+1}(F^s K^\bullet, W)\right)$ is mapped by $E_r^{p,q}(\alpha_s)$ into the vector space

$$ {}^{I}F_d^s E_r^{p,q}(K^\bullet, W) \cap \ker\left(d_r : E_r^{p,q}(K^\bullet, W) \longrightarrow E_r^{p+r,q-r+1}(K^\bullet, W)\right), $$

whose image in $E_{r+1}^{p,q}(K^\bullet, W)$ is by definition ${}^{I}G^s E_{r+1}^{p,q}(K^\bullet, W)$. As ${}^{I}F_d^s$ coincides with the image of α_s, the claim follows from the commutativity of the diagram above. □

By a dual argument, we can define a filtration $(E_{r+1}^{p,q}(K^\bullet, W), {}^{II}G)$ by

$$ {}^{II}G^s E_{r+1}^{p,q}(K^\bullet, W) := \mathrm{Im}\left(H({}^{II}F_d^s E_r^{p,q}(K^\bullet, W), d_r) \longrightarrow H(E_r^{p,q}(K^\bullet, W), d_r\right). $$

With the same arguments as before, but *reversing* all arrows, we can show also

$$ {}^{II}G^s E_{r+1}^{p,q}(K^\bullet, W) \subset {}^{II}F_d^s E_{r+1}^{p,q}(K^\bullet, W). $$

Now we are ready to prove (2) by induction. Let us define

$$ \mathcal{P}(r) \Leftrightarrow {}^{I}F_d E_r^{p,q}(K^\bullet, W) \subset F_i E_r^{p,q}(K^\bullet, W) \subset {}^{II}F_d E_r^{p,q}(K^\bullet, W). $$

From point (1) of this lemma we know that $\mathcal{P}(0)$ is true.

Assume now $\mathcal{P}(r)$. When we consider the cohomology of each term, we get the following chain of inclusions (note that inclusions are preserved by functoriality):

$$ \begin{array}{ccccc} {}^{I}GE_{r+1}^{p,q}(K^\bullet, W) & \subset & F_i E_{r+1}^{p,q}(K^\bullet, W) & \subset & {}^{II}GE_{r+1}^{p,q}(K^\bullet, W) \\ \cup & & & & \cap \\ {}^{I}F_d E_r^{p,q}(K^\bullet, W) & & & & {}^{II}F_d E_{r+1}^{p,q}(K^\bullet, W). \end{array} $$

Then also $\mathcal{P}(r+1)$ holds.

(3) We study now the situation for $r = \infty$. Recall from the general theory of spectral sequences that $E_\infty \cong E_r$ for $r \gg 0$ (because W is biregular) This allows us to define the first and the second direct filtration on $E_\infty^{p,q}(K^\bullet, W)$, in the

following way:

$$ {}^{I}F_d^s E_\infty^{p,q}(K^\bullet, W) = \mathrm{Im}(E_\infty^{p,q}(F^s K^\bullet, W) \longrightarrow E_\infty^{p,q}(K^\bullet, W)); $$

$$ {}^{II}F_d^s E_\infty^{p,q}(K^\bullet, W) = \ker(E_\infty^{p,q}(K^\bullet, W) \longrightarrow E_\infty^{p,q}(K^\bullet / F^s K^\bullet, W)). $$

Another general result is

$$ E_\infty^{p,q}(K^\bullet, W) \cong \mathrm{Gr}_{-p}^W H^{p+q}(K^\bullet). $$

This gives us the filtration F, defined as follows. For every index n, the morphism of complexes $F^s K^\bullet \longrightarrow K^\bullet$ induces a map $H^n(F^s K^\bullet) \longrightarrow H^n(K^\bullet)$ on cohomology. Then

$$ F^s H^n(K^\bullet) := \mathrm{Im}(H^n(F^s K^\bullet) \to H^n(K^\bullet)) $$

and the filtered space $(H^{p+q}(K^\bullet), F)$ induces $(E_\infty^{p,q}(K^\bullet, W), F)$, by considering $E_\infty^{p,q}(K^\bullet, W)$ as a subquotient of $H^{p+q}(K^\bullet)$.

Let us consider again the inclusion $(F^s K^\bullet, F) \hookrightarrow (K^\bullet, F)$. It induces a natural map

$$(E_\infty(F^s K^\bullet, W), F) \longrightarrow (E_\infty(F^s K^\bullet, W), F)$$

which is a filtered morphism. In particular, we have

$$\begin{tikzcd}
F^s E_\infty(F^s K^\bullet, W) \arrow[r] \arrow[d, equal] & E_\infty(F^s K^\bullet, W) \arrow[d, hook] \\
E_\infty(F^s K^\bullet, W) \arrow[r] \arrow[ur, dashed] & E_\infty(K^\bullet, W),
\end{tikzcd}$$

so that

$$^{I}F_d^s E_\infty(K^\bullet, W) = \mathrm{Im}(E_\infty(F^s K^\bullet, W) \longrightarrow E_\infty(K^\bullet, W)) \subset F^s E_\infty(K^\bullet, W).$$

The inclusion $F^s E_\infty(K^\bullet, W) \subset {}^{II}F_d^s E_\infty(K^\bullet, W)$ can be proved dually. $\square$

Theorem 4.19 (Deligne). *Let $(K^\bullet, W, F)$ be a differential complex endowed with 2 filtrations. In particular, suppose W to be biregular.*

1. *Suppose that for $r = 0, 1, \ldots, r_0$ the differentials d_r of the spectral sequence associated to W are strictly compatible with the inductive filtration. Then the filtrations ${}^{I}F_d, {}^{II}F_d, F_i$ coincide on $E_0, \ldots, E_{r_0+1}$.*

2. *Suppose that for all r the differentials d_r are strictly compatible with the inductive filtration on E_r. Then the four filtrations ${}^{I}F_d, {}^{II}F_d, F, F_i$ on E_∞ coincide.*

3. *If we assume that $\forall r \geq 0$ the differentials d_r are strictly compatible with the inductive filtration on E_r, then $E_r(K^\bullet, F)$ degenerates at E_1 and*

$$\begin{aligned}(\mathrm{Gr}_F^p(E_r(K^\bullet, W)), d_r) &\cong (E_r(\mathrm{Gr}_F^p(K^\bullet, W), d_r) \\ &\cong E_r(F^p K^\bullet, W)/E_r(F^{p+1} K^\bullet, W).\end{aligned}$$

Proof. (1) Let us consider $\forall r = 0, 1, \ldots$ the proposition

$$\mathcal{P}(r) \Leftrightarrow 0 \to E_r(F^s K^\bullet, W) \to E_r(K^\bullet, W) \to E_r(K^\bullet/F^s) \to 0 \text{ is exact } \forall s \in \mathbb{Z}.$$

$\mathcal{P}(0)$ holds, because on E_0 all filtrations coincide. Assume next that $\mathcal{P}(r)$ holds for some r, $0 \leq r \leq r_0$. This implies that the first and the second direct filtrations coincide on $E_r(K^\bullet, W)$, hence ${}^{I}F_d = {}^{II}F_d = F_i$ on $E_r(K^\bullet, W)$ for the Comparison Lemma (4.18). Denote this unique filtration by F. We have

$$\begin{aligned} E_r(F^s K^\bullet, W) &= F^s(E_r(K^\bullet, W)) \\ E_r(K^\bullet/F^s K^\bullet, W) &= E_r(K^\bullet, W)/F^s E_r(K^\bullet, W). \end{aligned}$$

By the compatibility of d_r with the inductive filtration, we get by Lemma 4.13 the following exact sequence:

$$\begin{array}{cl}
0 & \\
\downarrow & \\
H(F^s E_r(K^\bullet, W), d_r) & = H(E_r(F^s K^\bullet, W), d_r) = E_{r+1}(F^s K^\bullet, W) \\
\downarrow & \\
H(E_r(K^\bullet, W), d_r) & = E_{r+1}(K^\bullet, W) \\
\downarrow & \\
H(E_r(K^\bullet, W)/F^s E_r(K^\bullet, W), d_r) & = E_{r+1}(K^\bullet/F^s K^\bullet, W) \\
\downarrow & \\
0. &
\end{array}$$

Then also $\mathcal{P}(r+1)$ holds. Notice that the exactness of this sequence is a stronger property that the mere equality between the three filtrations.

(2) Assume that $\forall r \geq 0$, the differential $d_r(E_r(K^\bullet, W))$ is strictly compatible with the inductive filtration F_i. We know that $E_\infty(K^\bullet, W) \cong E_{r_0}(K^\bullet, W) \cong E_{r+1}(K^\bullet, W) \cong \cdots$ for some index r_0, because W is biregular. Then (1) implies ${}^I F_d = F_i = {}^{II} F_d$ on E_∞, and the claim follows from point (3) of the Comparison Lemma (4.18).

(3) Let us consider a fixed integer $r \geq 0$. By assumption, $d_r(E_r(K^\bullet, W)$ is strictly compatible with F_i. By (1) the two direct filtrations and the inductive filtration coincide on $E_r(K^\bullet, W)$. Denote by F this unique filtration.

It is a general fact that then $\forall p \geq 0$ also $(F^p K^\bullet, W, F)$ satisfies our hypotheses, and so does $(K^\bullet/F^{p+1}K^\bullet, W, F)$. This follows from the following lemma, whose proof is left as an exercise.

Lemma 4.20. *Let $(K^\bullet, W, F)$ be a differential complex endowed with two filtrations. Then, if $\forall r \geq 0$ the differential $d_r(K^\bullet, W)$ is strictly compatible with the inductive filtration induced by F on $E_r(K^\bullet, W)$, we have also that $d_r(F^a K^\bullet/F^b K^\bullet, W)$ is strictly compatible $\forall r \geq 0$ with the inductive filtration on $E_r(F^a K^\bullet/F^b K^\bullet, W, F)$, for any choice of $a \leq b$.*

This means that we can apply (1) to the exact sequence

$$0 \longrightarrow F^{p+1}K^\bullet \longrightarrow F^p K^\bullet \longrightarrow \operatorname{Gr}_F^p K^\bullet \longrightarrow 0$$

to get

$$0 \longrightarrow E_r(F^{p+1}K^\bullet, W) \longrightarrow E_r(F^p K^\bullet, W) \longrightarrow E_r(\operatorname{Gr}_F^p K^\bullet, W) \longrightarrow 0.$$

Analogously, from

$$0 \longrightarrow \operatorname{Gr}_F^p K^\bullet \longrightarrow K^\bullet/F^{p+1}K^\bullet \longrightarrow K^\bullet/F^p K^\bullet \longrightarrow 0$$

we obtain

$$0 \longrightarrow E_r(\operatorname{Gr}_F^p K^\bullet, W) \longrightarrow E_r(K^\bullet/F^{p+1}K^\bullet, W) \longrightarrow E_r(K^\bullet/F^p K^\bullet, W) \longrightarrow 0.$$

All this can be summarized by saying that the following diagram is commutative, and that all its rows, columns and skew columns are exact.

Diagram 4.21 (Deligne).

$$\begin{tikzcd}
 & 0 \arrow[dr] & & & & \\
0 & E_r(\mathrm{Gr}^p_F K^\bullet, W) \arrow[l] \arrow[dd, equal] & E_r(F^p K^\bullet, W) \arrow[l] \arrow[dr, "\beta"'] & & E_r(F^{p+1}K^\bullet, W) \arrow[ll] \arrow[dl, "\alpha"] & 0 \arrow[l] \\
 & & & E_r(K^\bullet, W) \arrow[dl, "\alpha'"'] \arrow[dr, "\beta'"] & & \\
0 \arrow[r] & E_r(\mathrm{Gr}^p_F K^\bullet, W) \arrow[r] & E_r(K^\bullet/F^{p+1}K^\bullet, W) \arrow[rr] \arrow[dl] & & E_r(K^\bullet/F^p K^\bullet, W) \arrow[r] & 0 \\
 & 0 & & & &
\end{tikzcd}$$

Let us extract some of the information provided by the diagram. First, by its commutativity we obtain

$$\begin{array}{ll}
\operatorname{Im}\beta & = {}^I F^p_d E_r(K^\bullet, W) \\
\cup & \\
\operatorname{Im}\alpha & = {}^I F^{p+1}_d E_r(K^\bullet, W) = F^{p+1} E_r(K^\bullet, W).
\end{array}$$

In the second place, from the exactness of the first row we have

$$E_r(\mathrm{Gr}^p_F K^\bullet, W) \cong E_r(F^p K^\bullet, W)/E_r(F^{p+1}K^\bullet, W),$$

and, from the injectivity of α and β,

$$\operatorname{Im}\alpha \cong E_r(F^{p+1}K^\bullet, W),$$

$$\operatorname{Im}\beta \cong E_r(F^p K^\bullet, W).$$

All this relations together imply

$$\mathrm{Gr}^p_F E_r(K^\bullet, W) = \operatorname{Im}\alpha / \operatorname{Im}\beta \cong E_r(\mathrm{Gr}^p_F K^\bullet, W).$$

Remark that also a dual point of view on the diagram is possible (taking into account $\ker\alpha', \ker\beta'$).

It remains to prove that $E_r(K^\bullet, F)$ degenerates at E_1. From point (1) we know that for all $0 \ll r < \infty$ the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & E_r(F^p K^\bullet, W) \arrow[r] \arrow[d, equal] & E_r(K^\bullet, W) \arrow[r] \arrow[d, equal] & E_r(K^\bullet/F^p K^\bullet, W) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E_\infty(F^p K^\bullet, W) \arrow[r] \arrow[d, equal] & E_\infty(K^\bullet, W) \arrow[r] \arrow[d, equal] & E_\infty(K^\bullet/F^p K^\bullet, W) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Gr}^W H(F^p K^\bullet) \arrow[r] & \mathrm{Gr}^W H(K^\bullet) \arrow[r] & \mathrm{Gr}^W H(K^\bullet/F^p K^\bullet) \arrow[r] & 0
\end{tikzcd}$$

transforms into an exact sequence of complexes. Apply Lemma 4.14 to the 3-terms complex

$$\{0 \longrightarrow H(F^pK^\bullet) \longrightarrow H(K^\bullet) \longrightarrow H(K^\bullet/F^pK^\bullet) \longrightarrow 0\} \qquad (*)$$

filtered by W. Then the sequence (*) is exact and, by Lemma 4.13, the spectral sequence $E_r(K^\bullet, F)$ degenerates at E_1. □

4.3.3 Proof of the Deligne's Fundamental Theorem (4.10).

(1) This is a consequence of the definitions. See also Remark 4.3.

(2a) is again a matter of definitions. Moreover, (2a) $\Rightarrow$(3).

(2b) We want to prove that the first differential d_1 of the spectral sequence of the filtered complex $(\mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{Q}}), W)$ is strictly compatible with the filtration induced by F.

As $(\mathrm{Gr}^W_{-p} \mathcal{K}^\bullet_{\mathbb{Q}}, (\mathrm{Gr}^W_{-p} \mathcal{K}^\bullet_{\mathbb{C}}, F), \mathrm{Gr}^W_{-p}(\beta))$ is a rational Hodge complex of sheaves of weight $-p$, we have that F induces a Hodge structure of weight $(p+q-p) = q$ on $E_1(\mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{C}}), W) = \mathbb{H}^{p+q}(X, \mathrm{Gr}^W_{-p} \mathcal{K}^\bullet_{\mathbb{C}})$. Explicitly, the Hodge filtration on $\mathbb{H}^{p+q}(X, \mathrm{Gr}^W_{-p} \mathcal{K}^\bullet_{\mathbb{C}})$ is

$$F^i\mathbb{H}^{p+q}(X, \mathrm{Gr}^W_{-p} \mathcal{K}^\bullet_{\mathbb{C}}) = \mathrm{Im}\left(\mathbb{H}^{p+q}(X, F^i \mathrm{Gr}^W_{-p} \mathcal{K}^\bullet_{\mathbb{C}}) \longrightarrow \mathbb{H}^{p+q}(X, \mathrm{Gr}^W_{-p} \mathcal{K}^\bullet_{\mathbb{C}})\right).$$

This is precisely the first direct filtration on $E_1(\mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{C}}), W)$. Recall that on E_1 the first direct filtration coincides with the second and with the inductive filtration.

We have that

$$d_1 : E_1^{p,q}(K^\bullet_C, W) \longrightarrow E_1^{p+1,q}(K^\bullet_{\mathbb{C}}, W)$$

is compatible with the Hodge filtration F, because this coincides with the first direct filtration, which is compatible with the differentials. Thus d_1 is defined over $\mathbb{Q}$ and preserves the Hodge filtration. Hence it is a morphism of Hodge structures, which is automatically compatible with the Hodge filtration.

(2c) We want to prove that the spectral sequence $E_r(\mathcal{R}\Gamma(Z, \mathcal{K}^\bullet_{\mathbb{Q}}), W)$ degenerates at E_2. This is a consequence of the following lemma.

Lemma 4.22. *For $r \geq 0$, the differentials d_r of the spectral sequence $E_r(K^\bullet_{\mathbb{C}}, W)$ are strictly compatible with the inductive filtration. More precisely, $d_r = 0$ for $r \geq 2$.*

(2d) and (2e) are consequences of Theorem 4.19. □

Proof of Lemma 4.22. For $r = 0$, we have $E_0(K^\bullet_{\mathbb{C}}, W) = \mathrm{Gr}^W_{-p} K^{p+q}_{\mathbb{C}}$. Then the claim follows from the fact that $(\mathrm{Gr}^W_{-p} K^\bullet_{\mathbb{Q}}, (\mathrm{Gr}^W_{-p} K^\bullet_{\mathbb{C}}, F), \mathrm{Gr}^W_{-p}(\beta))$ is a rational Hodge complex.

For $r = 1$, the claim coincides with (2b) of Theorem 4.10.

For $r \geq 2$, the situation is as follows. Since $E_1^{p,q}(K^\bullet_{\mathbb{C}}, W)$ is a Hodge structure of weight q, and d_1 is a morphism of Hodge structures, the subquotient

$E_2^{p,q}(K^\bullet_{\mathbb{C}}, W)$ endowed with the filtration F_i is a Hodge structure of weight q. As d_0 and d_1 are strictly compatible with F_i, by Theorem 4.19 all filtrations coincide on E_2. In particular, F_i is compatible with d_2, as the first and the second direct filtrations are compatible with the differentials. The differential d_2 is also rationally defined, so

$$d_2 : E_2^{p,q}(K^\bullet_{\mathbb{C}}, W) \longrightarrow E_2^{p+2,q-1}(K^\bullet_{\mathbb{C}}, W)$$

is a morphism of Hodge structures. But the weight of $E_2^{p,q}(K^\bullet_{\mathbb{C}}, W)$ is q, while the weight of $E_2^{p+2,q-1}(K^\bullet_{\mathbb{C}}, W)$ is $q-1$: then the only possibility for d_2 is $d_2 = 0$.

Analogous considerations can be applied to all E_r with $r \geq 3$. □

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